

# GLOBAL DYNAMICS ABOVE THE FIRST EXCITED ENERGY FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH A POTENTIAL

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**ABSTRACT.** Consider the focusing nonlinear Schrödinger equation (NLS) with a potential with a single negative eigenvalue. It has solitons with negative small energy, which are asymptotically stable, and solitons with positive large energy, which are unstable. We classify the global dynamics into 9 sets of solutions in the phase space including both solitons, restricted by small mass, radial symmetry, and an energy bound slightly above the second lowest one of solitons. The classification includes a stable set of solutions which start near the first excited solitons, approach the ground states locally in space for large time with large radiation to the spatial infinity, and blow up in negative finite time.

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## 1. INTRODUCTION

We continue from [5] the study of global dynamics for the nonlinear Schrödinger equation with a potential  $V = V(|x|) : \mathbb{R}^3 \rightarrow \mathbb{R}$  which decays as  $|x| \rightarrow \infty$ ,

$$i\dot{u} + Hu = |u|^2u, \quad H := -\Delta + V, \quad u(t, x) : \mathbb{R}^{1+3} \rightarrow \mathbb{C}, \quad (1.1)$$

in the case  $H$  has a bound state  $0 < \phi_0 \in L^2(\mathbb{R}^3)$

$$H\phi_0 = e_0\phi_0, \quad e_0 < 0, \quad \|\phi_0\|_2 = 1, \quad (1.2)$$

and no other eigenfunction nor resonance (so  $\phi_0$  is the ground state of  $H$ ). Henceforth,  $\|\cdot\|_p$  denotes the  $L^p(\mathbb{R}^3)$  norm. See Section 1.5 for the precise assumptions on  $V$ . As a simple case, it suffices to assume  $V = V(|x|) \in \mathcal{S}(\mathbb{R}^3)$  besides the above spectral condition.

In [5], the global behavior was investigated for all radial solutions  $u$  with small mass and energy below the first excited state. In this paper, the analysis goes slightly *above the threshold energy*:

$$\begin{aligned} \mathbb{M}(u) &:= \int_{\mathbb{R}^3} \frac{|u|^2}{2} dx \ll 1, \\ \mathbb{E}(u) &:= \int_{\mathbb{R}^3} \frac{|\nabla u|^2 + V|u|^2}{2} - \frac{|u|^4}{4} dx < \mathcal{E}_1(\mathbb{M}(u))(1 + \varepsilon^2), \end{aligned} \quad (1.3)$$

for some small  $\varepsilon > 0$ , where  $\mathcal{E}_1(\mu)$  denotes *the second lowest energy of solitons* for the prescribed mass  $\mathbb{M}(u) = \mu$ . The goal of this paper is to give a complete classification of global dynamics including both stable and unstable solitons, as well as scattering and blow-up, in a phase space restricted only by the conserved quantities and the symmetry. The main questions are which initial data  $u(0)$  lead to each type of solutions, and how the solution  $u$  can change its behavior from one type to another along its evolution. See [5, Introduction], [6] and references therein for more background and motivation of this setting.

**1.1. Solitons.** In order to state the main result precisely, we first need to define the energy levels of the ground state and the excited states. Consider the elliptic equation for the solution of the form  $u(t) = e^{-it\omega}\varphi(x)$  for any time frequency  $\omega \in \mathbb{R}$

$$(H + \omega)\varphi = |\varphi|^2\varphi \quad (1.4)$$

and let  $\mathcal{S}$  be the set of all radial solutions

$$\mathcal{S} := \{\varphi \in H_r^1(\mathbb{R}^3) \mid \exists \omega > 0, \text{ s.t. (1.4)}\}, \quad (1.5)$$

where  $H_r^1(\mathbb{R}^3)$  denotes the subspace of radially symmetric functions of  $H^1(\mathbb{R}^3)$  with the norm  $\|\varphi\|_{H^1}^2 = \|\nabla \varphi\|_2^2 + \|\varphi\|_2^2$ . The restriction to  $\omega > 0$  comes from the absence of embedded eigenvalue for the Schrödinger operator  $-\Delta + V - |\varphi|^2$ , which follows from the ODE in the radial setting. The ground state energy level is defined for each prescribed mass  $\mu > 0$  by

$$\mathcal{E}_0(\mu) := \inf\{\mathbb{E}(\varphi) \mid \varphi \in \mathcal{S}, \mathbb{M}(\varphi) = \mu\}, \quad (1.6)$$

and the  $j$ -th excited state energy level is defined inductively by

$$\mathcal{E}_j(\mu) := \inf\{\mathbb{E}(\varphi) \mid \varphi \in \mathcal{S}, \mathbb{M}(\varphi) = \mu, \mathbb{E}(\varphi) > \mathcal{E}_{j-1}(\mu)\} \quad (1.7)$$

together with the corresponding set of solitons

$$\mathcal{S}_j := \{\varphi \in \mathcal{S} \mid \mathbb{E}(\varphi) = \mathcal{E}_j(\mathbb{M}(\varphi))\}. \quad (1.8)$$

The small mass constraint  $\mathbb{M}(u) \ll 1$  enables us to identify the ground states  $\mathcal{S}_0$  as bifurcation of the linear ground state  $\phi_0$  for  $\omega \rightarrow -e_0 + 0$ , and the first excited states  $\mathcal{S}_1$  as rescaled perturbation for  $\omega \rightarrow \infty$  of the ground state  $Q$  of the nonlinear Schrödinger equation without the potential:

$$i\dot{u} - \Delta u = |u|^2 u. \quad (1.9)$$

More precisely, let  $Q \in H^1(\mathbb{R}^3)$  be the unique positive radial solution of

$$-\Delta Q + Q = Q^3. \quad (1.10)$$

There are constants  $0 < \mu_*, z_* \ll 1 \ll \omega_* < \infty$  and  $C^1$  maps

$$\begin{aligned} (\Phi, \Omega) : Z_* := \{z \in \mathbb{C} \mid |z| < z_*\} &\rightarrow H_r^1(\mathbb{R}^3) \times (-e_0, \infty) \\ \Psi : [\omega_*, \infty) &\rightarrow H_r^1(\mathbb{R}^3), \end{aligned} \quad (1.11)$$

such that  $(\varphi, \omega) = (\Phi[z], \Omega[z]), (\Psi[\omega], \omega)$  are solutions of (1.4) satisfying

$$\begin{aligned} \Phi[z] &= z\phi_0 + \gamma, \quad \gamma \perp \phi_0, \quad \|\gamma\|_{H^1} \lesssim |z|^3, \quad \Omega[z] = -e_0 + O(|z|^2), \\ \Psi[\omega](|x|) &= \omega^{1/2}(Q + \gamma)(\omega^{1/2}x), \quad \|\gamma\|_{H^1} \lesssim \omega^{-1/4}, \end{aligned} \quad (1.12)$$

with asymptotic formulas of mass and energy

$$\begin{aligned} \mathbb{M}(\Phi[z]) &= |z|^2/2 + O(|z|^6), \quad \mathbb{E}(\Phi[z]) = e_0|z|^2/2 + O(|z|^4), \\ \mathbb{M}(\Psi[\omega]) &= \omega^{-1/2}\mathbb{M}(Q) + O(\omega^{-3/4}), \quad \mathbb{E}(\Psi[\omega]) = \omega^{1/2}\mathbb{E}(Q) + O(\omega^{1/4}), \end{aligned} \quad (1.13)$$

as well as their monotonicity  $\frac{d}{da}\mathbb{M}(\Phi[a]) \sim a$ ,  $\frac{d}{d\omega}\mathbb{M}(\Psi[\omega]) \sim -\omega^{-3/2}$ , and

$$\mathcal{S}_0|_{\mathbb{M} < \mu_*} = \{\Phi[z] \mid z \in Z_*\}, \quad \mathcal{S}_1|_{\mathbb{M} < \mu_*} = \{e^{i\theta}\Psi[\omega] \mid \theta \in \mathbb{R}, \omega > \omega_*\}. \quad (1.14)$$

Moreover, as  $\mu_* > \mu \rightarrow +0$ ,

$$\begin{aligned} \mathcal{E}_0(\mu) &= e_0\mu(1 + O(\mu)), \\ \mathcal{E}_1(\mu) &= \mathbb{M}(Q)^2\mu^{-1}(1 + O(\mu^{1/2})), \\ \mathcal{E}_2(\mu) &> 4\mathbb{M}(Q)^2\mu^{-1}. \end{aligned} \quad (1.15)$$

Note that the energy for (1.9)

$$\mathbb{E}^0(\varphi) := \int_{\mathbb{R}^3} \frac{|\nabla \varphi|^2}{2} - \frac{|\varphi|^4}{4} dx \quad (1.16)$$

is identical to  $\mathbb{M}(\varphi)$  if  $\varphi$  is a solution of (1.10). A proof of the above statements is given in [2, Lemma 2.1] for the ground state part, and in Lemma 2.6 for the excited state part.

**1.2. Types of behavior.** In this paper, we consider the following three types of behavior of the solution  $u$ , both in positive time and in negative time, which leads to a classification into 9 non-empty sets of solutions.

- (1) Scattering to the ground states  $\mathcal{S}_0$ .
- (2) Blow-up.
- (3) Trapping by the first excited states  $\mathcal{S}_1$ .

All the solutions below the excited states  $\mathbb{E}(u) < \mathcal{E}_1(\mathbb{M}(u))$  are completely split into (1) and (2) with the same behavior in  $t > 0$  and in  $t < 0$ , which is explicitly predictable by the initial data, using the virial functional:

$$\mathbb{K}_2(u) := \partial_{\alpha=1} \mathbb{E}(\alpha^{3/2} u(\alpha x)) = \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{r V_r |u|^2}{2} - \frac{3|u|^4}{4} dx, \quad (1.17)$$

where  $r := |x|$  is the radial variable. See [5, Theorem 1.1] for the precise statement. The difference between below and above the excited energy are the new type (3), and solutions with different types of behavior in  $t > 0$  and in  $t < 0$ , namely *transition* among (1)–(3).

The following are precise definitions for (1)–(3), under the small mass constraint. Let  $u$  be a solution of (1.1). The local wellposedness in  $H^1(\mathbb{R}^3)$  implies that the maximal existence interval  $(T_-(u), T_+(u)) \subset \mathbb{R}$  is uniquely defined such that

$$u \in C((T_-(u), T_+(u)); H^1(\mathbb{R}^3)) \quad (1.18)$$

solves (1.1) for  $T_-(u) < t < T_+(u)$ .

We say that  $u$  *blows up* in  $t > 0$ , if  $T_+(u) < \infty$ . Otherwise, we say that  $u$  *is global* in  $t > 0$ . We say that  $u$  *scatters to the ground states* (or *scatters to  $\Phi$*  in short) as  $t \rightarrow \infty$ , if for some  $C^1$  function  $z : (T_-(u), \infty) \rightarrow Z_*$  and  $u_+ \in H_r^1(\mathbb{R}^3)$

$$\lim_{t \rightarrow \infty} \|u(t) - \Phi[z(t)] - e^{-it\Delta} u_+\|_{H^1(\mathbb{R}^3)} = 0. \quad (1.19)$$

For each  $\omega > 0$ , we introduce the following equivalent norm of  $H^1(\mathbb{R}^3)$

$$\|\varphi\|_{H_\omega^1}^2 := \int_{\mathbb{R}^3} \omega^{-1/2} |\nabla \varphi|^2 + \omega^{1/2} |\varphi|^2 dx. \quad (1.20)$$

It dominates the homogeneous Sobolev norm  $\dot{H}^{1/2}$  uniformly, and is the appropriate rescaling for the first excited states, because

$$\|\Psi[\omega]\|_{H_\omega^1} = \|Q\|_{H^1} (1 + O(\omega^{-1/4})). \quad (1.21)$$

The open neighborhood of the small-mass part of  $\mathcal{S}_1$  within distance  $\delta > 0$  in this metric is denoted by

$$\mathcal{N}_\delta(\Psi) := \{\varphi \in H_r^1(\mathbb{R}^3) \mid \exists \theta \in \mathbb{R}, \exists \omega > \omega_*, \|e^{i\theta} \Psi[\omega] - \varphi\|_{H_\omega^1} < \delta\}. \quad (1.22)$$

We say that  $u$  is trapped by the first excited states (or trapped by  $\Psi$  in short) as  $t \rightarrow \infty$ , if  $u(t) \in \mathcal{N}_\delta(\Psi)$  for large  $t$  and some small fixed  $0 < \delta \ll 1$ .

We can easily distinguish the above three types using the  $L_x^4$  norm for small mass solutions as follows. If  $u$  scatters to  $\Phi$  as  $t \rightarrow \infty$ , then

$$\|u(t)\|_4 = \|\Phi[z]\|_4 + o(1) \sim \|\Phi[z]\|_2 + o(1) \lesssim \sqrt{\mathbb{M}(u)} + o(1) \ll 1, \quad (1.23)$$

as  $t \rightarrow \infty$ . If  $u$  blows up in  $t > 0$ , then as  $t \rightarrow T_+(u)$ ,

$$\begin{aligned} \|u(t)\|_4^4/2 &= -2\mathbb{E}(u) + \int_{\mathbb{R}^3} |\nabla u(t)|^2 + V|u|^2 dx \\ &\geq -2\mathbb{E}(u) + \|\nabla u(t)\|_2^2 + O(\|u(t)\|_2^2 + \|u(t)\|_4^2) \rightarrow \infty, \end{aligned} \quad (1.24)$$

since  $V \in (L^2 + L^\infty)(\mathbb{R}^3)$ . If  $u$  is trapped by  $\Psi$  as  $t \rightarrow \infty$ , then for large  $t$ ,

$$\|u(t)\|_4 = \|\Psi[\omega]\|_4 - O(\delta\omega^{1/8}) \sim \omega^{1/8} \gg 1, \quad (1.25)$$

where  $\Psi[\omega]$  is estimated by (2.30)-(2.31) and the remainder is bounded by Gagliardo-Nirenberg

$$\|\varphi\|_4 \lesssim \|\nabla \varphi\|_2^{3/4} \|\varphi\|_2^{1/4} \lesssim \omega^{1/8} \|\varphi\|_{H_\omega^1}. \quad (1.26)$$

**1.3. The main result.** We consider the following set of initial data where the mass is bounded and the energy is allowed to exceed the first excited state slightly:

$$\mathcal{H}^{\mu,\varepsilon} := \{\varphi \in H_r^1(\mathbb{R}^3) \mid \mathbb{M}(\varphi) < \mu, \mathbb{E}(\varphi) < \mathcal{E}_1(\mathbb{M}(\varphi))(1 + \varepsilon^2)\}. \quad (1.27)$$

These initial data can be classified by behavior of the solution in  $t > 0$ :

$$\begin{aligned} \mathcal{S} &:= \{u(0) \in H_r^1(\mathbb{R}^3) \mid u \text{ scatters to } \Phi \text{ as } t \rightarrow \infty\}, \\ \mathcal{B} &:= \{u(0) \in H_r^1(\mathbb{R}^3) \mid u \text{ blows up in } t > 0\}, \\ \mathcal{T}_\delta &:= \{u(0) \in H_r^1(\mathbb{R}^3) \mid u(t) \in \mathcal{N}_\delta(\Psi) \text{ for large } t\}, \end{aligned} \quad (1.28)$$

where  $u$  is the solution of (1.1) for the initial data  $u(0)$ . The same classification for  $t < 0$  is given by their complex conjugate, thanks to the time inversion symmetry: if  $u(t, x)$  is a solution of (1.9), then so is  $\bar{u}(-t, x)$ .

**Theorem 1.1.** *If  $\mu$  and  $\varepsilon$  are small enough, then we have the following.*

$$\mathcal{H}^{\mu,\varepsilon} \subset \mathcal{S} \cup \mathcal{B} \cup \mathcal{T}_\delta, \quad (1.29)$$

for some  $\delta \leq C\varepsilon$ , where  $C > 0$  is some constant independent of  $\mu, \varepsilon > 0$ . Each of the 9 combinations of dynamics in positive and negative time

$$\mathcal{H}^{\mu,\varepsilon} \cap \mathcal{X} \cap \overline{\mathcal{Y}}, \quad \mathcal{X}, \mathcal{Y} \in \{\mathcal{S}, \mathcal{B}, \mathcal{T}_\delta\}, \quad (1.30)$$

contain infinitely many orbits.  $\mathcal{S}$  is open.  $\mathcal{T}_\delta \cap \mathcal{H}^{\mu,\varepsilon}$  is a  $C^1$  manifold of codimension 1 in  $H_r^1(\mathbb{R}^3)$ , connected and unbounded.  $\mathcal{T}_\delta \cap \overline{\mathcal{T}_\delta} \cap \mathcal{H}^{\mu,\varepsilon}$  is a  $C^1$  manifold of codimension 2, connected and contained in  $\mathcal{N}_{\delta'}(\Psi)$  for some  $\delta' \leq C\delta$ . There is a connected open neighborhood of  $\mathcal{T}_\delta \cap \mathcal{H}^{\mu,\varepsilon}$  which is separated by  $\mathcal{T}_\delta$  into two connected open sets contained respectively in  $\mathcal{S}$  and in  $\mathcal{B}$ .  $\overline{\mathcal{T}_\delta} \cap \mathcal{H}^{\mu,\varepsilon}$  is also separated by  $\mathcal{T}_\delta$  into two manifolds contained respectively in  $\mathcal{S}$  and in  $\mathcal{B}$ .

**1.4. Notation.** First recall some notation in [5].  $L^p$ ,  $B_{p,q}^s$ , and  $H_p^s$  denote respectively the Lebesgue, inhomogeneous Besov and Sobolev spaces on  $\mathbb{R}^3$ , and  $H^s := H_2^s$ .  $\dot{H}^s = \dot{H}_2^s$  denotes the homogeneous Sobolev space, and  $\|\cdot\|_p$  denotes the  $L^p$  norm on  $\mathbb{R}^3$ .  $\mathcal{S}'(\mathbb{R}^3)$  denotes the space of tempered distributions. The complex-valued and the real-valued  $L^2$  inner products are denoted respectively by  $(f|g) := \int_{\mathbb{R}^3} f(x)\overline{g(x)}dx$  and  $\langle f|g \rangle := \operatorname{Re}(f|g)$ . For any Banach function space  $X$  on  $\mathbb{R}^3$ , its subspace of radial functions is denoted by  $X_r$ , the space equipped with the weak topology is denoted by  $w\text{-}X$ , the weak limit is denoted by  $w\text{-}\lim$ , and  $L_t^p X$  denotes the  $L^p$  space for  $t \in \mathbb{R}$  with values in  $X$ . Some standard Strichartz norms on  $\mathbb{R}^{1+3}$  are denoted by

$$\operatorname{Stz}^s := L_t^\infty H^s \cap L_t^2 B_{6,2}^s, \quad \mathbf{st} := L_t^4 L^6. \quad (1.31)$$

For any function space  $Z$  on  $\mathbb{R}^{1+3}$  and a set  $I \subset \mathbb{R}$ , the restriction of  $Z$  onto  $I \times \mathbb{R}^3$  is denoted by  $Z(I)$ . For any  $w : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $\varphi \in H^1$ , the following define some functionals

$$\begin{aligned} \lceil w \rceil(\varphi) &:= \frac{1}{2} \langle w\varphi | \varphi \rangle, \quad \mathbb{M} := \lceil 1 \rceil, \quad \mathbb{G}(\varphi) := \frac{1}{4} \|\varphi\|_4^4, \\ \mathbb{H}^0(\varphi) &:= \frac{1}{2} \|\nabla \varphi\|_2^2, \quad \mathbb{E}^0 := \mathbb{H}^0 - \mathbb{G}, \quad \mathbb{E} := \mathbb{E}^0 + \lceil V \rceil. \end{aligned} \quad (1.32)$$

For any  $p \in (0, \infty]$ ,  $t \in \mathbb{R}$  and  $\varphi \in \mathcal{S}'(\mathbb{R}^3)$ , the  $L^p$  preserving dilation and its generator are denoted by

$$\mathcal{S}_p^t \varphi(x) := e^{3t/p} \varphi(e^t x), \quad \mathcal{S}_p' F(\varphi) := \lim_{t \rightarrow 0} \frac{F(\mathcal{S}_p^t \varphi) - F(\varphi)}{t}, \quad (1.33)$$

for any  $F$  acting on functions on  $\mathbb{R}^3$ . We have  $\mathcal{S}_p' \varphi = (x \cdot \nabla + 3/p)\varphi$ . Then we define

$$\mathbb{K}_2 := \mathcal{S}_2' \mathbb{E} = 2\mathbb{H}^0 - 3\mathbb{G} - \lceil \mathcal{S}_\infty' V \rceil. \quad (1.34)$$

Next, some new notation and symbols are introduced. For any symbols  $F, X, Y$ , the difference is denoted by (this is a slight modification from [5])

$$\triangleleft F(X^\triangleright, Y^\triangleright, \dots) := F(X^\perp, Y^\perp, \dots) - F(X^\natural, Y^\natural, \dots), \quad (1.35)$$

where the symbols  $\triangleright, \perp, \natural$  and  $\triangleleft$  are reserved for this purpose, and the underline is to avoid confusion with exponents. The subspace and the projection orthogonal (in the real sense) to  $\varphi \in \mathcal{S}(\mathbb{R}^3)$  are denoted by

$$\varphi^\perp := \{\psi \in \mathcal{S}'(\mathbb{R}^3) \mid \langle \varphi | \psi \rangle = 0\}, \quad P_\varphi^\perp := 1 - \|\varphi\|_2^{-2} \varphi \langle \varphi |. \quad (1.36)$$

The projection to the continuous spectral subspace of  $H$  is denoted by

$$P_c := P_{\phi_0}^\perp P_{i\phi_0}^\perp = 1 - \phi_0(\phi_0|. \quad (1.37)$$

For two Banach spaces  $X$  and  $Y$ , the Banach space of bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{B}(X, Y)$ . For any  $\omega > 0$  and  $\varphi \in H^1(\mathbb{R}^3)$ , denote

$$\begin{aligned} \mathbb{A}_\omega &:= \mathbb{E} + \omega \mathbb{M}, \quad \mathbb{K}_{0,\omega}(\varphi) := \langle \mathbb{A}'_\omega(\varphi) | \varphi \rangle = 2(\mathbb{A}_\omega - \mathbb{G})(\varphi), \\ \mathbb{E}^\omega(\varphi) &:= \omega^{-1/2} \mathbb{E}(\mathbb{S}_\omega^{-1} \varphi) = (\mathbb{E}^0 + \lceil V^\omega \rceil)(\varphi), \\ \mathbb{A}^\omega(\varphi) &:= \omega^{-1/2} \mathbb{A}_\omega(\mathbb{S}_\omega^{-1} \varphi) = (\mathbb{E}^\omega + \mathbb{M})(\varphi), \\ \mathbb{K}_2^\omega(\varphi) &:= \omega^{-1/2} \mathbb{K}_2(\mathbb{S}_\omega^{-1} \varphi) = (2\mathbb{H}^0 - 3\mathbb{G} - \lceil \mathcal{S}'_\infty V^\omega \rceil)(\varphi), \\ \mathbb{J}^\omega &:= \mathbb{A}^\omega - \frac{1}{2} \mathbb{K}_2^\omega = \frac{1}{2} \mathbb{G} + \frac{1}{2} \lceil \mathcal{S}'_{3/2} V^\omega \rceil + \mathbb{M}, \end{aligned} \tag{1.38}$$

where the rescaling operator  $\mathbb{S}_\omega$  and the rescaled potential  $V^\omega$  are defined by

$$\mathbb{S}_\omega \varphi(x) := \omega^{-1/2} \varphi(\omega^{-1/2} x), \quad V^\omega(x) := \omega^{-1/2} \mathbb{S}_\omega V, \tag{1.39}$$

and the version without the potential

$$\mathbb{A} := \mathbb{E}^0 + \mathbb{M} = \lim_{\omega \rightarrow \infty} \mathbb{A}^\omega. \tag{1.40}$$

In this paper, most of the analysis will be done in the variables rescaled by  $\mathbb{S}_\omega$ , where the smallness of  $\mathbb{M}$  corresponds to the largeness of  $\omega$ . This is to avoid getting large scaling factors in the estimates around the first excited states, and to make the formulations similar in the leading order to the case without the potential. Of course, we still need to take care of the long-time impact of the potential and the ground states, even if they are small in some sense.

**1.5. Assumptions on the potential.** In addition to the assumptions on  $V$  in [5], we assume that  $V \in L^2(\mathbb{R}^3)$ . Hence the precise list of assumptions on  $V$  is

- (i)  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  is radially symmetric.
- (ii)  $V \in L^2(\mathbb{R}^3) \cap |x|L^1(\mathbb{R}^3)$  and  $x\nabla V, x^2\nabla^2 V \in (L^2 + L^\infty)(\mathbb{R}^3)$ .
- (iii)  $-\Delta + V$  on  $L^2_r(\mathbb{R}^3)$  has a unique and negative eigenvalue, denoted by  $e_0$ .
- (iv) The wave operator  $\lim_{t \rightarrow \infty} e^{itH} e^{it\Delta}$  and its adjoint are bounded on  $H^k_{\mathfrak{p}}(\mathbb{R}^3)$  for some  $\mathfrak{p} > 6$  and  $k = 0, 1$ ,

where  $L^\infty_0(\mathbb{R}^3) := \{\varphi \in L^\infty(\mathbb{R}^3) \mid \lim_{R \rightarrow \infty} \|\varphi\|_{L^\infty(|x| > R)} = 0\}$ . For example, if  $V_0$  is a radial positive Schwartz function on  $\mathbb{R}^3$ , then there exist  $b > a > 0$  such that  $V = -cV_0$  satisfies the above assumptions for  $a < c < b$ . See [5] for more comments.  $V$  is fixed throughout the paper, so that some “constants” can depend on  $V$ .

## 2. THE FIRST EXCITED STATE AND THE LINEARIZED OPERATOR

In this section, we analyze the first excited state for small mass and the spectrum of the linearized operator around it.

**2.1. Zero-mass asymptotic of energy.** For any solution  $\varphi \in H^1_r$  of (1.4) with  $\mathbb{M}(\varphi) \ll 1$ , we may apply the small mass dichotomy [5, Lemma 2.3], since  $\mathbb{K}_2(\varphi) = \langle \mathbb{A}'_\omega(\varphi) | \mathcal{S}'_2 \varphi \rangle = 0$ . Then we have either  $\|\varphi\|_{H^1} \sim \|\varphi\|_2 \ll 1$  or

$$\mathbb{G}(\varphi) \gtrsim \mathbb{H}^0(\varphi) \gtrsim \mathbb{M}(\varphi)^{-1} \gg 1. \tag{2.1}$$

If  $\mathbb{M}(\varphi)$  is small enough, then the former case implies that  $\varphi = \Phi[z]$  for some  $z \in Z_*$ , see [5, Propositions 2.4 and 2.5]. Hence we may concentrate on the latter case (2.1) for excited states. Since  $V, \mathcal{S}'_p V \in L^2 + L^\infty$ , we have [5, Lemma 2.1]

$$|[V](\varphi)| + |[\mathcal{S}'_p V](\varphi)| \leq \varepsilon \sqrt{\mathbb{G}(\varphi)} + C_{p,\varepsilon} \mathbb{M}(\varphi), \quad (2.2)$$

for any  $\varepsilon, p > 0$ . Combining this, (2.1) and  $\mathbb{K}_2(\varphi) = 0 = \mathbb{K}_{0,\omega}(\varphi)$ , we obtain approximate Pohozaev identities as  $\mu := \mathbb{M}(\varphi) \rightarrow 0$ ,

$$\mathbb{E}(\varphi) = \omega \mathbb{M}(\varphi) + o(\mu^{-1/2}) = \frac{1}{2} \mathbb{G}(\varphi) + o(\mu^{-1/2}) = \frac{1}{3} \mathbb{H}^0(\varphi) + o(\mu^{-1/2}) \gtrsim \mu^{-1}, \quad (2.3)$$

and so  $\omega \gtrsim \mu^{-2}$ . Then rescaling the solution by  $\varphi_\omega := S_\omega \varphi = \omega^{-1/2} \varphi(\omega^{-1/2} x)$  leads to the rescaled equation with time frequency 1

$$(-\Delta + V^\omega + 1)\varphi_\omega = |\varphi_\omega|^2 \varphi_\omega, \quad V^\omega(x) = \omega^{-1} V(\omega^{-1/2} x), \quad (2.4)$$

and rescaled functionals

$$(\mathbb{H}^0, \mathbb{M}, \mathbb{G}, \mathbb{A}^\omega)(\varphi_\omega) = \omega^{-1/2} (\mathbb{H}^0, \omega \mathbb{M}, \mathbb{G}, \mathbb{A}_\omega)(\varphi) \sim \omega^{1/2} \mu \gtrsim 1. \quad (2.5)$$

The rescaled potential is small in the following sense. Decompose  $V = W_2 + W_\infty$  such that  $\|W_2\|_2 + \|W_\infty\|_\infty = \|V\|_{L^2+L^\infty}$  and rescale  $W_p^\omega := \omega^{-1/2} S_\omega W_p$ . Then

$$\begin{aligned} |[V^\omega](\varphi)| &\leq \frac{1}{2} [\|W_2^\omega\|_2 \|\varphi\|_4^2 + \|W_\infty^\omega\|_\infty \|\varphi\|_2^2] \\ &= \|V\|_{L^2+L^\infty} [\omega^{-1/4} \sqrt{\mathbb{G}(\varphi)} + \omega^{-1} \mathbb{M}(\varphi)] \\ &\lesssim \omega^{-1/4} \|\nabla \varphi\|_2^{3/2} \|\varphi\|_2^{1/2} + \omega^{-1} \|\varphi\|_2^2 \lesssim \omega^{-1/4} \|\varphi\|_{H^1}^2, \end{aligned} \quad (2.6)$$

where the third inequality is by Gagliardo-Nirenberg. We have the same estimates on  $\mathcal{S}'_p V^\omega$  and  $\mathcal{S}'_p \mathcal{S}'_q V^\omega$ . In particular, for large  $\omega$ ,

$$\mathbb{A}^\omega(\varphi) - \frac{1}{3} \mathbb{K}_2^\omega(\varphi) = \frac{1}{6} \|\nabla \varphi\|_2^2 + \frac{1}{2} \|\varphi\|_2^2 + \frac{1}{3} [\mathcal{S}'_1 V^\omega](\varphi) \sim \|\varphi\|_{H^1}^2. \quad (2.7)$$

The small mass dichotomy [5, Lemma 2.3] is rewritten for the rescaled function.

**Lemma 2.1.** *There exists a constant  $C_D \in (1, \infty)$  such that for any  $\omega \in (0, \infty)$  and  $\varphi \in H^1(\mathbb{R}^3)$  satisfying*

$$C_D \mathbb{K}_2^\omega(\varphi) < \mathbb{M}(\varphi)^{-1} \quad \text{and} \quad C_D \mathbb{M}(\varphi) < \omega^{1/2}, \quad (2.8)$$

*we have one of the following (2.9)–(2.11)*

$$\mathbb{H}^0(\varphi) \leq C_D \omega^{-1} \mathbb{M}(\varphi), \quad \mathbb{G}(\varphi) \leq C_D \omega^{-3/2} \mathbb{M}(\varphi)^2. \quad (2.9)$$

$$\omega^{-1} \mathbb{M}(\varphi) \leq C_D \mathbb{H}^0(\varphi) \leq C_D^2 \mathbb{K}_2^\omega(\varphi) \leq C_D^3 \mathbb{H}^0(\varphi). \quad (2.10)$$

$$\mathbb{M}(\varphi)^{-1} \leq C_D \mathbb{H}^0(\varphi) \leq C_D^2 \mathbb{G}(\varphi). \quad (2.11)$$

*For any  $p, q > 0$ , there is a constant  $C_{p,q} > 0$  such that in the case (2.11),*

$$|[V^\omega](\varphi)| + |[\mathcal{S}'_p V^\omega](\varphi)| + |[\mathcal{S}'_p \mathcal{S}'_q V^\omega](\varphi)| \leq C_{p,q} (\omega^{-1/2} \mathbb{M}(\varphi))^{1/2} \mathbb{H}^0(\varphi). \quad (2.12)$$

*If  $\varphi_n$  is in the case (2.11) for all  $n$ , weakly converging in  $H_r^1$  as  $n \rightarrow \infty$  and  $\liminf_{n \rightarrow \infty} \mathbb{K}_2^\omega(\varphi_n) \leq 0$ , then the weak limit is also in the case (2.11).*



*Proof.* The estimate on  $\mathbb{G}$  in the case (2.9) follows by Gagliardo-Nirenberg. The left side of (2.12) is bounded using (2.6) by

$$(\beta^{1/4} + \beta)\mathbb{H}^0(\varphi) \lesssim \beta^{1/4}\mathbb{H}^0(\varphi), \quad (2.13)$$

where  $\beta := \omega^{-1}\mathbb{H}^0(\varphi)^{-1}\mathbb{M}(\varphi) \leq C_D(\omega^{-1/2}\mathbb{M}(\varphi))^2 < 1/C_D$ . Thus we obtain (2.12).

Suppose that  $\varphi_n \in H_r^1$  are all in the case (2.11),  $\varphi_n \rightarrow \varphi$  weakly in  $H^1$  and  $\liminf_{n \rightarrow \infty} \mathbb{K}_2^\omega(\varphi_n) \leq 0$ , then  $\varphi_n \rightarrow \varphi$  strongly in  $L^4$ . The assumption (2.8) is preserved by the weak limit, and  $\mathbb{K}_2^\omega(\varphi) \leq \liminf_{n \rightarrow \infty} \mathbb{K}_2^\omega(\varphi_n) \leq 0$ . Since  $\varphi_n$  is in the case (2.11),

$$\omega^{-1/2} < C_D^{-1}\mathbb{M}(\varphi_n)^{-1} \leq C_D\mathbb{G}(\varphi_n) \rightarrow C_D\mathbb{G}(\varphi). \quad (2.14)$$

In particular,  $\varphi \neq 0$ . Since  $\mathbb{K}_2^\omega(\varphi) \leq 0$ , (2.10) is impossible. Since (2.9) implies

$$C_D\mathbb{G}(\varphi) \leq \omega^{-3/2}C_D^2\mathbb{M}(\varphi)^2 < \omega^{-1/2}, \quad (2.15)$$

contradicting (2.14), hence we have (2.11). The rest of the lemma follows simply by rescaling [5, Lemma 2.3].  $\square$

Now consider any sequence of  $(\varphi, \omega)$  solving (1.4) such that  $\mu := \mathbb{M}(\varphi) \rightarrow 0$  and  $\varphi \notin \Phi[Z_*]$ . Then (2.5) implies that boundedness of the rescaled functions  $\varphi_\omega := S_\omega\varphi$  in  $L^2(\mathbb{R}^3)$  is equivalent to boundedness in  $L^4(\mathbb{R}^3)$ , boundedness in  $\dot{H}^1(\mathbb{R}^3)$ ,

$$\omega \sim \mu^{-1/2}, \quad (2.16)$$

and weak convergence in  $H^1(\mathbb{R}^3)$  of  $\varphi_\omega$  along a subsequence. In that case, let  $\varphi_\infty \in H_r^1(\mathbb{R}^3)$  be the weak limit along the subsequence. Then it solves the static nonlinear Schrödinger equation without the potential

$$(-\Delta + 1)\varphi_\infty = |\varphi_\infty|^2\varphi_\infty, \quad (2.17)$$

and the weak convergence implies that the limit energy satisfies

$$\mathbb{A}(\varphi_\infty) \leq \liminf \mathbb{A}(\varphi_\omega) = \liminf \omega^{-1/2}\mathbb{A}_\omega(\varphi) = \liminf 2\sqrt{\mathbb{M}\mathbb{E}(\varphi)}, \quad (2.18)$$

where the last two equalities follow from (2.5) and (2.3).

By the classical results on radial solutions of (2.17), all the solutions  $\psi$  are real-valued (modulo complex rotation  $\psi \mapsto e^{i\theta}\psi$ ), satisfying  $\mathbb{E}^0(\psi) = \mathbb{M}(\psi)$ . The least energy non-trivial solution is the unique positive solution, namely the ground state  $Q$ . The other radial solutions have at least one zero point in  $r = |x| > 0$ , and those  $\psi \neq 0$  with  $m$  zeros have at least  $m + 1$  times energy:

$$\mathbb{A}(\psi) > (m + 1)\mathbb{A}(Q) = 2(m + 1)\mathbb{M}(Q) > 0. \quad (2.19)$$

The asymptotic (2.18) of energy implies that if  $(\varphi, \omega)$  is a sequence of solutions to (1.4) such that  $\mathbb{M}(\varphi) \rightarrow 0$ ,  $\varphi \notin \Phi[Z_*]$ , and  $\mathbb{A}^\omega(S_\omega\varphi) \leq 2\mathbb{A}(Q)$ , then we have  $e^{i\theta}\varphi \rightarrow Q$  for some sequence of  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ , strongly in  $H_r^1(\mathbb{R}^3)$ . The convergence has to be strong, since otherwise

$$0 = 2(\mathbb{A} - \mathbb{G})(Q) \leq \liminf \omega^{-1/2}\mathbb{K}_{0,\omega}(\varphi) = 0 \quad (2.20)$$

would become a strict inequality. Thus we have obtained

**Lemma 2.2.** *For any  $\delta > 0$ , there exists  $\mu(\delta) > 0$  such that if  $(\varphi, \omega) \in H_r^1 \times (0, \infty)$  satisfies the soliton equation (1.4) and*

$$\mathbb{M}(\varphi) \leq \mu(\delta), \quad \mathbb{A}^\omega(S_\omega \varphi) \leq 2\mathbb{A}(Q), \quad (2.21)$$

*then either  $\varphi \in \Phi[Z_*]$  or*

$$\|S_\omega \varphi - e^{i\theta} Q\|_{H^1} < \delta \quad (2.22)$$

*for some  $\omega \sim \mathbb{M}(\varphi)^{-2}$  and  $\theta \in \mathbb{R}$ .*

The first excited states satisfy the above energy constraint [5, Proposition 2.5]. In the next subsection, we prove that they are indeed the only solitons satisfying (2.22) for small mass. Then the above lemma implies the estimate on  $\mathcal{E}_2$  in (1.15).

**2.2. Construction of the first excited state.** The above lemma allows us to expand the first excited state in the rescaled variables around  $Q$ . For that purpose, consider the linearized operators for (2.17) around  $Q$

$$\mathcal{L}v := L_+ v_1 + iL_- v_2, \quad L_+ := -\Delta + 1 - 3Q^2, \quad L_- := -\Delta + 1 - Q^2, \quad (2.23)$$

where  $v_1 := \operatorname{Re} v$  and  $v_2 := \operatorname{Im} v$  for any  $v \in \mathcal{S}'(\mathbb{R}^3)$ . The null space of  $\mathcal{L}$  on  $L_r^2$  equals to  $\operatorname{span}\{iQ\}$ .  $\mathcal{L}$  is invertible on the radial subspace orthogonal to  $iQ$ , and  $\mathcal{L}^{-1}$  is bounded  $H_r^{-1} \cap (iQ)^\perp \rightarrow H_r^1 \cap (iQ)^\perp$ . In other words,  $(L_+)^{-1} : H_r^{-1} \rightarrow H_r^1$  and  $(L_-)^{-1} : H_r^{-1} \cap Q^\perp \rightarrow H_r^1 \cap Q^\perp$  are bounded.

Let  $\omega > 0$  and let  $\psi \in H_r^1(\mathbb{R}^3)$  be a solution of (2.4) close to the ground states of (2.17), in other words

$$\delta := \inf_{\theta \in \mathbb{R}} \|\psi - e^{i\theta} Q\|_{H^1} \quad (2.24)$$

is small enough. Then there is a unique  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$  such that

$$\psi = e^{i\theta}(Q + v) \implies 0 = \langle iQ | v \rangle = \langle iQ | e^{-i\theta} \psi \rangle, \quad \|v\|_{H^1} \sim \delta. \quad (2.25)$$

Indeed, it is explicitly given by  $\theta = \arg(\psi|Q)$ . Then (2.4) is rewritten into the following equation for  $v$ :

$$\mathcal{L}v = N(v) - V^\omega(Q + v), \quad N(v) := 2Q|v|^2 + Qv^2 + |v|^2v. \quad (2.26)$$

and, since  $v \perp iQ$ ,

$$v = (\mathcal{L}|_{(iQ)^\perp})^{-1} P_{iQ}^\perp (N(v) - V^\omega(Q + v)), \quad (2.27)$$

where the orthogonal projection  $P_{iQ}^\perp : \mathcal{S}' \rightarrow (iQ)^\perp$  is bounded on  $H^s$  for any  $s \in \mathbb{R}$ . Using Sobolev, Hölder and (2.6), we have

$$\begin{aligned} \|N(v) - V^\omega(Q + v)\|_{H^{-1}} &\lesssim \|v\|_{H^1}^2 + \|v\|_{H^1}^3 + \omega^{-1/4}(\|Q\|_{H^1} + \|v\|_{H^1}) \\ &\lesssim \omega^{-1/4}(1 + \|v\|_{H^1}) + \|v\|_{H^1}^2 + \|v\|_{H^1}^3, \end{aligned} \quad (2.28)$$

and similarly for any small  $v^0, v^\perp \in H^1$ ,

$$\|\triangleleft[N(v^\triangleright) - V^\omega(Q + v^\triangleright)]\|_{H^{-1}} \lesssim (\omega^{-1/4} + \|v^0\|_{H^1} + \|v^\perp\|_{H^1}) \|\triangleleft v^\triangleright\|_{H^1}. \quad (2.29)$$

Hence the right hand side of (2.27) is a contraction map for small  $v \in H_r^1$  if  $\omega \gg 1$ , having a unique fixed point  $v \in H_r^1 \cap (iQ)^\perp$  which is small in  $H^1$ . Thus

$$\Psi[\omega] := S_\omega^{-1} Q_\omega, \quad Q_\omega := Q + v, \quad (2.30)$$

is a family of solutions to (1.4), smoothly depending on  $\omega \gg 1$  with

$$v = -(L_+)^{-1}V^\omega Q + O(\omega^{-1/2}) = O(\omega^{-1/4}) \text{ in } H_r^1(\mathbb{R}^3). \quad (2.31)$$

Denote the orbit of the rescaled soliton by

$$\mathcal{Q}_\omega := \{e^{i\theta}Q_\omega \mid \theta \in \mathbb{R}\}. \quad (2.32)$$

For the energy and mass, we deduce from (2.30)–(2.31)

$$\begin{aligned} \mathcal{E}_1(\mathbb{M}(\Psi[\omega])) &= \mathbb{E}(\Psi[\omega]) = \omega^{1/2}(\mathbb{E}^0(Q) + O(\omega^{-1/4})), \\ \mathbb{M}(\Psi[\omega]) &= \omega^{-1/2}(\mathbb{M}(Q) + O(\omega^{-1/4})). \end{aligned} \quad (2.33)$$

Hence putting  $\mu = \mathbb{M}(\Psi[\omega])$ , we obtain

$$\omega = \mu^{-2}(\mathbb{M}(Q)^2 + O(\mu^{1/2})), \quad \mathcal{E}_1(\mu) = \mu^{-1}(\mathbb{M}\mathbb{E}^0(Q) + O(\mu^{1/2})). \quad (2.34)$$

Since the ground states  $\Phi[Z_*]$  have smaller  $\omega \sim -e_0$ , these solitons have the least energy for fixed  $\omega \gg 1$ . Hence they are also the constrained minimizers

$$\mathbb{A}_\omega(\Psi[\omega]) = \inf\{\mathbb{A}_\omega(\varphi) \mid 0 \neq \varphi \in H_r^1(\mathbb{R}^3), \mathbb{K}_{0,\omega}(\varphi) = 0\}. \quad (2.35)$$

In particular,  $\Psi[\omega] > 0$  on  $\mathbb{R}^3$ . For the analysis of dynamics, it is more important to relate it to the virial identity or  $\mathbb{K}_2$ .

**Lemma 2.3.** *There is a constant  $C_M \in (1, \infty)$  such that if  $\omega \in (1, \infty]$  is large enough then we have the following for any  $\varphi \in H_r^1(\mathbb{R}^3)$ .*

- (1)  $\mathbb{M}(\varphi) < 2\mathbb{A}(Q)$  and (2.9)  $\implies (\mathbb{M} + \omega\mathbb{H}^0)(\varphi) < C_M$ .
- (2) (2.11)  $\implies C_S\mathbb{M}\mathbb{H}^0(\varphi) \geq 1 \implies (\mathbb{M} + \omega\mathbb{H}^0)(\varphi) > C_M$ , where

$$C_S := \max(C_D, 2 + (\mathbb{M}\mathbb{H}^0(Q))^{-1}). \quad (2.36)$$

- (3) If  $(\mathbb{M} + \omega\mathbb{H}^0)(\varphi) \leq C_M$  then the solution  $u$  of (1.1) with  $u(0) = S_\omega^{-1}\varphi$  scatters to  $\Phi$  as  $t \rightarrow \pm\infty$ , satisfying  $\mathbb{H}^0(S_\omega u(t)) < \mathbb{H}^0(Q)/4$  for all  $t \in \mathbb{R}$ .
- (4)  $\mathcal{Q}_\omega = \{e^{i\theta}Q_\omega\}_\theta$  is the set of minimizers of

$$\begin{aligned} &\inf\{\mathbb{A}^\omega(\varphi) \mid 0 \neq \varphi \in H_r^1, \mathbb{K}_2^\omega(\varphi) = 0, (\mathbb{M} + \omega\mathbb{H}^0)(\varphi) > C_M\} \\ &= \inf\{\mathbb{J}^\omega(\varphi) \mid 0 \neq \varphi \in H_r^1, \mathbb{K}_2^\omega(\varphi) \leq 0, (\mathbb{M} + \omega\mathbb{H}^0)(\varphi) > C_M\}. \end{aligned} \quad (2.37)$$

The above choice (2.36) of the constant  $C_S$  will be used later for scaling invariant separation between the ground states and the excited states.

*Proof.* Since (2.9) and  $\mathbb{M}(\varphi) < 2\mathbb{A}(Q)$  imply  $(\mathbb{M} + \omega\mathbb{H}^0)(\varphi) \leq (1 + C_D)\mathbb{M}(\varphi) < 2(1 + C_D)\mathbb{A}(Q)$ , taking  $C_M \geq 2(1 + C_D)\mathbb{A}(Q)$  yields (1). (2) follows from

$$2\sqrt{\mathbb{M}\mathbb{H}^0} \leq \omega^{-1/2}(\mathbb{M} + \omega\mathbb{H}^0), \quad (2.38)$$

and taking  $\omega$  so large that  $\omega^{-1/2}C_M < 2C_S^{-1/2}$ . A similar condition yields (3), because of  $\|S_\omega^{-1}\varphi\|_{H^1}^2 = 2\omega^{-1/2}(\mathbb{M} + \omega\mathbb{H}^0)(\varphi)$ ,  $\mathbb{H}^0(S_\omega u) = \omega^{-1/2}\mathbb{H}^0(u)$  and the asymptotic stability [2] of the ground states in  $H^1$ .

If  $\omega = \infty$ , then the constraint on  $\mathbb{M} + \omega\mathbb{H}^0$  in (2.37) becomes trivial and (4) is reduced to a well known statement for the NLS without potential. So we may restrict (4) to the case  $\omega < \infty$  (though the argument is essentially the same).

For  $\omega$  large enough,  $Q_\omega$  satisfies all the constraints in (2.37). To show the equality in (2.37), it suffices to show that  $\mathbb{J}^\omega(\varphi)$  is bigger than the first line of (2.37) for any  $\varphi$  satisfying the constraints and  $\mathbb{K}_2^\omega(\varphi) < 0$ , since  $\mathbb{J}^\omega = \mathbb{A}^\omega - \mathbb{K}_2^\omega/2$ .

Suppose that  $\mathbb{J}^\omega(\varphi)$  is close to the second infimum and  $\mathbb{K}_2^\omega(\varphi) < 0$ . Then by (2.7)  $< \mathbb{J}^\omega(\varphi)$  we deduce that  $\mathbb{M}(\varphi) < 2\mathbb{A}(Q) < C_D^{-1}\omega^{1/2}$  for large  $\omega$ . Then  $(\mathbb{M} + \omega\mathbb{H}^0)(\varphi) > C_M$  and (1) preclude (2.9), while  $\mathbb{K}_2^\omega(\varphi) < 0$  precludes (2.10). Hence we have (2.11) by Lemma 2.1.

Consider the  $L^2$ -invariant scaling  $v(t) := \mathcal{S}_2^t \varphi$ , starting from  $t = 0$  and decreasing. As long as  $\mathbb{K}_2^\omega(v(t)) \leq 0$ , Lemma 2.1 applies to  $v(t)$ , and (1)-(2) with the continuity of  $v$  in  $t$  imply that  $v(t)$  stays in the case (2.11). Meanwhile, we have, using (2.12),

$$\begin{aligned} 2\mathcal{S}_2' \mathbb{J}^\omega(v) &= 3\mathbb{G}(v) - [\mathcal{S}_\infty' \mathcal{S}_{3/2}' V^\omega](v) \sim \mathbb{G}(v) \gtrsim 1, \\ \mathcal{S}_2' \mathbb{K}_2^\omega &= 2\mathbb{K}_2^\omega - 2\mathcal{S}_2' \mathbb{J}^\omega \lesssim -\mathbb{G}(v) \lesssim -1. \end{aligned} \quad (2.39)$$

Hence at some  $t < 0$ , we have  $\mathbb{J}^\omega(v(t)) < \mathbb{J}^\omega(\varphi)$ ,  $\mathbb{K}_2^\omega(v(t)) = 0$  and (2.11) for  $v(t)$ , so  $(\mathbb{M} + \omega\mathbb{H}^0)(v(t)) > C_M$  by (2). This implies the equality in (2.37).

Next we prove the existence of minimizer. Take any sequence  $\varphi_n \in H_r^1$  satisfying  $\varphi_n \neq 0$ ,  $\mathbb{K}_2^\omega(\varphi_n) = 0$ ,  $(\mathbb{M} + \omega\mathbb{H}^0)(\varphi_n) > C_M$  and  $\mathbb{A}^\omega(\varphi_n) \rightarrow (2.37)$ . The same argument as above implies that  $\varphi_n$  is in the case of (2.11). Passing to a subsequence, we have  $\varphi_n \rightarrow \exists \varphi$  weakly in  $H_r^1$ , then  $\mathbb{K}_2^\omega(\varphi) \leq 0$ ,  $\mathbb{J}^\omega(\varphi) \leq (2.37)$ , and by Lemma 2.1,  $\varphi$  also satisfies (2.11), so  $(\mathbb{M} + \omega\mathbb{H}^0)(\varphi) > C_M$  by (2). Hence  $\varphi$  is a minimizer of the second line of (2.37).

For any minimizer  $\varphi$  of (2.37), there is a Lagrange multiplier  $\mu \in \mathbb{R}$  such that  $(\mathbb{A}^\omega)'(\varphi) = \mu(\mathbb{K}_2^\omega)'(\varphi)$ . Then

$$0 = \mathbb{K}_2^\omega(\varphi) = \langle (\mathbb{A}^\omega)'(\varphi) | \mathcal{S}_2' \varphi \rangle = \mu \langle (\mathbb{K}_2^\omega)'(\varphi) | \mathcal{S}_2' \varphi \rangle = \mu \mathcal{S}_2' \mathbb{K}_2^\omega(\varphi) \quad (2.40)$$

together with  $\mathcal{S}_2' \mathbb{K}_2^\omega(\varphi) \neq 0$  by (2.39) implies  $\mu = 0$ . Therefore  $\varphi$  is a solution of (2.4), satisfying  $\mathbb{G}(\varphi) \sim 1 \gtrsim \mathbb{M}(\varphi)$  and  $\mathbb{A}^\omega(\varphi) \leq \mathbb{A}^\omega(Q_\omega)$ . Then Lemma 2.2 implies that  $\varphi \in \mathcal{Q}_\omega$  if  $\omega$  is large enough.  $\square$

**2.3. Rescaled linearization and spectrum.** Next we consider the linearization of (1.1) around the first excited soliton in the rescaled variables. Let

$$\mathcal{L}^\omega v := L_+^\omega v_1 + iL_-^\omega v_2, \quad \begin{cases} L_+^\omega := -\Delta + 1 + V^\omega - 3Q_\omega^2, \\ L_-^\omega := -\Delta + 1 + V^\omega - Q_\omega^2, \end{cases} \quad (2.41)$$

where  $Q_\omega$  is the rescaled excited state as in (2.30). The linearized operator for the evolution is given by  $i\mathcal{L}^\omega$  in the rescaled variables. The asymptotics (2.30)–(2.31) of  $\Psi[\omega]$  together with the smallness (2.6) of  $V^\omega$  implies

$$\mathcal{L}^\omega = \mathcal{L} + O(\omega^{-1/4}) \text{ in } \mathcal{B}(H^1, H^{-1}). \quad (2.42)$$

The gauge invariance for  $e^{i\theta} \times$  implies the trivial null direction

$$i\mathcal{L}^\omega iQ_\omega = 0. \quad (2.43)$$

Another direction comes from  $\omega$ . Differentiating the equation (1.4) for  $\Psi[\omega]$  yields

$$(H + \omega - 3\Psi[\omega]^2)\Psi'[\omega] = -\Psi[\omega], \quad (2.44)$$

which is rescaled to

$$L_+^\omega Q'_\omega = -Q_\omega, \quad (2.45)$$

where  $Q'_\omega \in H_r^1(\mathbb{R}^3)$  is defined by

$$Q'_\omega := S_\omega \omega \Psi'[\omega] = \frac{1}{2} \mathcal{S}'_3 Q_\omega + \omega \partial_\omega Q_\omega. \quad (2.46)$$

Remark that  $Q'_\omega \neq \partial_\omega Q_\omega$ . The above equation (2.45) is equivalent to

$$i\mathcal{L}^\omega Q'_\omega = -iQ_\omega \quad (2.47)$$

Since  $L_+^\omega = L_+ + O(\omega^{-1/4})$  in  $\mathcal{B}(H^1, H^{-1})$  and  $L_+ : H_r^1 \rightarrow H_r^{-1}$  is invertible,  $L_+^\omega$  is also invertible with

$$\|(L_+^\omega)^{-1} - (L_+)^{-1}\|_{\mathcal{B}(H_r^{-1}, H_r^1)} \lesssim \omega^{-1/4}. \quad (2.48)$$

Thus we obtain

$$Q'_\omega = (L_+^\omega)^{-1}(-Q_\omega) = (L_+^\omega)^{-1}(-Q + O(\omega^{-1/4})) = Q' + O(\omega^{-1/4}) \text{ in } H_r^1, \quad (2.49)$$

where

$$Q' := \frac{1}{2} \mathcal{S}'_3 Q = -(L_+)^{-1} Q \in H_r^1(\mathbb{R}^3). \quad (2.50)$$

This also tells us asymptotic formulas for  $\mathcal{E}_1''$  as follows. Since  $\mathbb{A}'_\omega = 0$  on solitons, putting  $\mu = \mathbb{M}(\Psi[\omega])$  we have

$$\mathcal{E}'_1(\mu) = -\omega = -\mu^{-2}(\mathbb{M}(Q)^2 + O(\mu^{1/2})), \quad \mathcal{E}_1''(\mu) = -\frac{d\omega}{d\mu}, \quad (2.51)$$

where the last term is computed by

$$\begin{aligned} \frac{d\mu}{d\omega} &= \langle \Psi[\omega] | \Psi'[\omega] \rangle = \omega^{-3/2} \langle Q_\omega | Q'_\omega \rangle \\ &= \omega^{-3/2} (\langle Q | Q' \rangle + O(\omega^{-1/4})) = -\omega^{-3/2} (\mathbb{M}(Q)/2 + O(\omega^{-1/4})). \end{aligned} \quad (2.52)$$

Therefore, for small  $\mu > 0$ ,

$$\mathcal{E}_1''(\mu) = \omega^{3/2} (2/\mathbb{M}(Q) + O(\omega^{-1/4})) = \mu^{-3} (2\mathbb{M}(Q)^2 + O(\mu^{1/2})) > 0. \quad (2.53)$$

Using  $\mathbb{E}^0(Q) = \mathbb{M}(Q)$ , the above formulas can also be written as

$$\mathcal{E}'_1(\mu) = -\mu^{-2} (\mathbb{M} \mathbb{E}^0(Q) + O(\mu^{1/2})), \quad \mathcal{E}_1''(\mu) = 2\mu^{-3} (\mathbb{M} \mathbb{E}^0(Q) + O(\mu^{1/2})). \quad (2.54)$$

Next we look for a pair of positive and negative eigenvalues. In the limit  $\omega \rightarrow \infty$ , we have some  $\alpha \in (0, \infty)$  and  $g_\pm \in \mathcal{S}_r(\mathbb{R}^3)$  satisfying

$$i\mathcal{L}g_\pm = \pm \alpha g_\pm, \quad g_- = \overline{g_+}, \quad \alpha \langle ig_+ | g_- \rangle = 2, \quad \langle iQ | g_+ \rangle > 0, \quad (2.55)$$

cf. [7]. Put  $g_\pm = g_1 \pm ig_2$ . Consider the eigenvalue problem

$$i\mathcal{L}^\omega g^\omega = \alpha_\omega g^\omega, \quad (2.56)$$

in the form

$$\alpha_\omega = \alpha(1 + c), \quad g^\omega = g_+ + \gamma, \quad \langle i\gamma | g_- \rangle = 0, \quad (2.57)$$

and  $|c| + \|\gamma\|_{H^1} = o(1)$  as  $\omega \rightarrow \infty$ , where  $c \in \mathbb{R}$  and  $\gamma \in H_r^1$  also depend on  $\omega$ . Putting  $R := i\mathcal{L}^\omega - i\mathcal{L}$ , the above equation (2.56) is equivalent to

$$(i\mathcal{L} - \alpha)\gamma = (-R + \alpha c)(g_+ + \gamma), \quad (2.58)$$

while the orthogonality yields an equation for  $c$

$$\begin{aligned} 0 &= \langle ig^\omega | (i\mathcal{L} + \alpha)g_- \rangle = \langle (i\mathcal{L} - \alpha)g^\omega | ig_- \rangle = \langle (i\mathcal{L} - \alpha)\gamma | ig_- \rangle \\ &= -2c - \langle R(g_+ + \gamma) | ig_- \rangle. \end{aligned} \quad (2.59)$$

Injecting it into the previous equation yields an equation for  $\gamma$  by itself

$$(i\mathcal{L} - \alpha)\gamma = (-R + \alpha \langle iR(g_+ + \gamma) | g_- \rangle / 2)(g_+ + \gamma) =: \mathcal{R}(\gamma), \quad (2.60)$$

and the above computation for  $c$  implies that  $\langle i\mathcal{R}(\gamma) | g_- \rangle = 0$  if  $\langle i\gamma | g_- \rangle = 0$ .

Since  $\|R\|_{\mathcal{B}(H_r^1, H_r^{-1})} \lesssim \omega^{-1/4}$ , we have

$$\|\mathcal{R}(\gamma)\|_{H^{-1}} \lesssim \omega^{-1/4}(1 + \|\gamma\|_{H^1})^2, \quad (2.61)$$

as well as a similar estimate for the difference. Hence (2.60) has a unique fixed point  $\gamma \in H_r^1 \cap (ig_-)^\perp$  for  $\omega \gg 1$ , provided that  $(i\mathcal{L} - \alpha)$  has a bounded inverse. Indeed

**Lemma 2.4.**  *$(i\mathcal{L} - \alpha)$  has a bounded inverse  $H^{-1} \rightarrow H^1$  on  $(ig_-)^\perp$ . More precisely, for any  $h \in H^{-1}(\mathbb{R}^3) \cap (ig_-)^\perp$ , there exists a unique  $f \in H^1(\mathbb{R}^3) \cap (ig_-)^\perp$  such that  $(i\mathcal{L} - \alpha)f = h$ , and moreover  $\|f\|_{H^1} \lesssim \|h\|_{H^{-1}}$ .*

*Proof.* First remark that  $\text{Ker}(i\mathcal{L} \mp \alpha) = \text{span}\{g_\pm\}$  follows from the fact that  $L_- \geq 0$  and  $L_+$  has only one negative eigenvalue. Indeed, if  $(i\mathcal{L} - \alpha)g = 0$  for some  $g = g_1 + ig_2 \in H^1(\mathbb{R}^3)$ , then  $\langle L_+g_1 | g_1 \rangle = -\langle L_-g_2 | g_2 \rangle < 0$  and  $g_2 = L_+g_1/\alpha$ , hence such a function  $g$  should live in one dimensional subspace, because of the spectrum of  $L_+$ .

The free operator  $i\mathcal{L}_0 - \alpha := i(1 - \Delta) - \alpha$  is invertible

$$(i\mathcal{L}_0 - \alpha)^{-1} = -[(1 - \Delta)^2 + |\alpha|^2]^{-1}[i(1 - \Delta) + \alpha] \quad (2.62)$$

which can be written as a Fourier multiplier, and bounded  $H^{-1} \rightarrow H^1$ . Moreover,  $i\mathcal{L} - \alpha = (i\mathcal{L}_0 - \alpha)(I + K)$ , where the operator  $K$  is defined by

$$K\varphi := (i\mathcal{L}_0 - \alpha)^{-1}Q^2(\varphi_2 - 3i\varphi_1), \quad (2.63)$$

and compact on  $H^1$ , hence  $\text{Ran}(I + K) = \text{Ker}(I + K^*)^\perp$ . Noting that

$$(i\mathcal{L} - \alpha)^* = -\mathcal{L}i - \alpha = i(i\mathcal{L} + \alpha)i, \quad (2.64)$$

we have  $\text{Ker}(I + K^*) = (i\mathcal{L}_0 - \alpha)^*i \text{Ker}(i\mathcal{L} + \alpha) = \text{span}\{(i\mathcal{L}_0 - \alpha)^*ig_-\}$ , and so

$$\text{Ran}(I + K) = (i\mathcal{L}_0 - \alpha)^{-1}(H^{-1} \cap (ig_-)^\perp). \quad (2.65)$$

Since  $g_+ \notin X := H^1 \cap (ig_-)^\perp$ , we have  $H^1 = X \oplus \text{span}\{g_+\}$ . This and  $\text{Ker}(I + K) = \text{span}\{g_+\}$  imply that  $I + K$  is bijective  $X \rightarrow \text{Ran}(I + K)$ . Hence the equation  $(i\mathcal{L} - \alpha)f = h$  has the unique solution

$$f = (I + K)|_X^{-1}(i\mathcal{L}_0 - \alpha)^{-1}h \in X \quad (2.66)$$

together with the boundedness  $\|f\|_{H^1} \lesssim \|(i\mathcal{L}_0 - \alpha)^{-1}h\|_{H^1} \lesssim \|h\|_{H^{-1}}$ .  $\square$

Thus we have obtained a pair of eigenfunctions for  $\omega \gg 1$

$$(i\mathcal{L}^\omega \mp \alpha_\omega)g_\pm^\omega = 0, \quad g_\pm^\omega = g_1^\omega \pm ig_2^\omega, \quad \alpha_\omega \langle ig_+^\omega | g_-^\omega \rangle = 2, \quad (2.67)$$

satisfying  $(\alpha_\omega, g_\pm^\omega) = (\alpha, g_\pm)(1 + O(\omega^{-1/4}))$  in  $\mathbb{R} \times H_r^1(\mathbb{R}^3)$ . The eigenfunction  $g_+^\omega$  is not exactly the above  $g^\omega$ , but it is normalized by a factor  $1 + O(\omega^{-1/4})$  to realize the last identity of (2.67).

In using the virial identity around  $Q_\omega$ , we will need that  $\langle (\mathbb{K}_2^\omega)'(Q_\omega) | g_1^\omega \rangle > 0$ , which follows from  $\langle iQ | g_+ \rangle = \langle Q | g_2 \rangle > 0$ . Indeed,

$$\begin{aligned} \langle (\mathbb{K}_2^\omega)'(Q_\omega) | g_1^\omega \rangle &= \langle (L_+^\omega/2 + 3L_-^\omega/2 - 2 - \mathcal{S}'_{3/2}V^\omega)Q_\omega | g_1^\omega \rangle \\ &= \alpha_\omega \langle Q_\omega | g_2^\omega \rangle / 2 - \langle \mathcal{S}'_{3/2}V^\omega Q_\omega | g_1^\omega \rangle = \alpha \langle Q | g_2 \rangle + O(\omega^{-1/4}) > \alpha \langle Q | g_2 \rangle / 2 > 0, \end{aligned} \quad (2.68)$$

if  $\omega$  is large enough.

**2.4. Expansion of the rescaled energy.** Using the linearized operator and its spectral decomposition, we can expand

$$\mathbb{A}^\omega(e^{i\theta}(Q_\omega + v)) = \mathbb{A}^\omega(Q_\omega) + \frac{1}{2}\langle \mathcal{L}^\omega v | v \rangle - C^\omega(v), \quad (2.69)$$

for  $v \in H_r^1$ , where the cubic and quartic terms are collected into

$$C^\omega(v) := \langle |v|^2 v | Q_\omega \rangle + \mathbb{G}(v) = O(\|v\|_{H^1}^3). \quad (2.70)$$

Expand  $v$  by the eigenfunctions of  $i\mathcal{L}^\omega$

$$v = b_+g_+^\omega + b_-g_-^\omega + \zeta = b_1g_1^\omega + b_2g_2^\omega + \zeta, \quad (2.71)$$

where  $b_\pm, b_1, b_2 \in \mathbb{R}$  are defined by

$$\begin{aligned} b_\pm &:= P_\pm^\omega v := \pm \alpha_\omega \langle iv | g_\mp^\omega \rangle / 2, \\ b_1 &:= P_1^\omega v := -\alpha_\omega \langle v_1 | g_2^\omega \rangle = b_+ + b_-, \\ b_2 &:= P_2^\omega v := -\alpha_\omega \langle v_2 | g_1^\omega \rangle = b_+ - b_-, \end{aligned} \quad (2.72)$$

such that

$$\zeta := P_c^\omega v := v - (P_+^\omega v)g_+^\omega - (P_-^\omega v)g_-^\omega \implies 0 = \langle i\zeta | g_\pm^\omega \rangle = \langle \zeta_1 | g_2^\omega \rangle = \langle \zeta_2 | g_1^\omega \rangle. \quad (2.73)$$

Then using  $(i\mathcal{L}^\omega \mp \alpha_\omega)g_\pm^\omega = 0$  and  $\alpha_\omega \langle ig_+^\omega | g_-^\omega \rangle = 2\alpha_\omega \langle g_1^\omega | g_2^\omega \rangle = 2$ , we obtain

$$\begin{aligned} \mathbb{A}^\omega(Q_\omega + v) - \mathbb{A}^\omega(Q_\omega) &= -2b_+b_- + \frac{1}{2}\langle \mathcal{L}^\omega \zeta | \zeta \rangle - C^\omega(v) \\ &= \frac{1}{2}[-b_1^2 + b_2^2 + \langle \mathcal{L}^\omega \zeta | \zeta \rangle] - C^\omega(v). \end{aligned} \quad (2.74)$$

If  $\varphi \in H_r^1(\mathbb{R}^3)$  is close to the rescaled excited states

$$d_{0,\omega}(\varphi) := \inf_{\theta \in \mathbb{R}} \|\varphi - e^{i\theta}Q_\omega\|_{H^1} \ll 1, \quad (2.75)$$

for some  $\omega$ , then there exists a unique  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$  such that

$$\varphi = e^{i\theta}(Q_\omega + v) \implies 0 = \langle iQ'_\omega | v \rangle = \langle iQ'_\omega | e^{-i\theta}\varphi \rangle, \quad \|v\|_{H^1} \sim d_{0,\omega}(\varphi). \quad (2.76)$$

Indeed, it is explicitly given by  $\theta = \arg(\varphi | Q'_\omega)$ , well-defined in the region  $\langle \varphi | Q'_\omega \rangle > 0$ . This orthogonality is inherited by the radiation component  $\zeta = P_c^\omega v$ , since  $\langle iQ'_\omega | g_\pm^\omega \rangle = 0$ . The energy controls  $P_c^\omega v$  through the following coercivity.

**Lemma 2.5.** *There exists a constant  $C \in [1, \infty)$  such that for large  $\omega$  and any  $\varphi \in H_r^1(\mathbb{R}^3)$ , we have*

$$\|\varphi\|_{H^1}^2/C \leq \langle \mathcal{L}^\omega \varphi | \varphi \rangle + C \langle \varphi_1 | g_2^\omega \rangle^2 + C \langle \varphi_2 | Q'_\omega \rangle^2 \leq C^2 \|\varphi\|_{H^1}^2. \quad (2.77)$$

*Proof.* In the limit case  $\omega = \infty$ , namely for NLS without the potential, this is [7, Lemma 2.2]. The above estimate is just a perturbation of that, since  $\mathcal{L}^\omega - \mathcal{L} = O(\omega^{-1/4})$  in  $\mathcal{B}(H^1, H^{-1})$ ,  $g_2^\omega = g_2 + O(\omega^{-1/4})$  and  $Q'_\omega = Q' + O(\omega^{-1/4})$  in  $H^1$ . Injecting these asymptotics into the limit estimate yields

$$\begin{aligned} \|\varphi\|_{H^1}^2 &\lesssim \langle \mathcal{L} \varphi | \varphi \rangle + C \langle \varphi_1 | g_2 \rangle^2 + C \langle \varphi_2 | Q' \rangle^2 \\ &= \langle \mathcal{L}^\omega \varphi | \varphi \rangle + C \langle \varphi_1 | g_2^\omega \rangle^2 + C \langle \varphi_2 | Q'_\omega \rangle^2 + O(\omega^{-1/4} \|\varphi\|_{H^1}^2), \end{aligned} \quad (2.78)$$

and then the left estimate of (2.77) after the last term is absorbed by the left side, while the other estimate of (2.77) is trivial.  $\square$

In view of the expansion (2.74), it is natural to introduce the following norm

$$\begin{aligned} \|v\|_\omega^2 &:= \frac{1}{2} \left[ (P_1^\omega v)^2 + (P_2^\omega v)^2 + \langle Q'_\omega | v_2 \rangle^2 + \langle \mathcal{L}^\omega P_\epsilon^\omega v | P_\epsilon^\omega v \rangle \right] \\ &= (P_+^\omega v)^2 + (P_-^\omega v)^2 + \frac{1}{2} \langle iQ'_\omega | v \rangle^2 + \frac{1}{2} \langle \mathcal{L}^\omega P_\epsilon^\omega v | P_\epsilon^\omega v \rangle, \end{aligned} \quad (2.79)$$

which is equivalent to the  $H^1$  norm on the radial subspace  $H_r^1$ , uniformly in  $\omega \gg 1$ . Using this norm, the expansion is rewritten as

$$\mathbb{A}^\omega(e^{i\theta}(Q_\omega + v)) = \mathbb{A}^\omega(Q_\omega) - b_1^2 + \|v\|_\omega^2 - C^\omega(v), \quad (2.80)$$

for any  $v$  satisfying the orthogonality

$$v \in \mathcal{V}_\omega := \{\varphi \in H_r^1(\mathbb{R}^3) \mid \langle iQ'_\omega | v \rangle = 0\}. \quad (2.81)$$

Similarly, the orthogonal subspace for  $P_\epsilon^\omega v$  is denoted by

$$\mathcal{Z}_\omega := \{\zeta \in H_r^1(\mathbb{R}^3) \mid 0 = \langle i\zeta | Q'_\omega \rangle = \langle i\zeta | g_\pm^\omega \rangle\}. \quad (2.82)$$

The following lemma is a summary of this section.

**Lemma 2.6.** *There are constants  $\mu_*, z_* \in (0, 1)$ ,  $\omega_* \in (1, \infty)$ , and  $C^1$  maps  $(\Phi, \Omega), \Psi$  satisfying (1.11)–(1.14) and the following. For  $\omega \geq \omega_*$ , we have*

$$|[V^\omega](\varphi)| + |[S'_\infty V^\omega](\varphi)| < \frac{1}{10} \|\varphi\|_{H^1}^2, \quad (2.83)$$

and (1)–(4) of Lemma 2.3. In particular,  $\mathcal{Q}_\omega = \{e^{i\theta} Q_\omega\}$  is the set of minimizers for (2.37), where  $Q_\omega = S_\omega \Psi[\omega] = Q + O(\omega^{-1/4})$  in  $H^1$ . More specifically,

$$\mathbb{A}^\omega(Q_\omega) < \frac{11}{10} \mathbb{A}(Q), \quad \text{MIH}^0(Q_\omega) < \frac{11}{10} \text{MIH}^0(Q), \quad \mathbb{E}^\omega(Q_\omega) > \frac{1}{2} \mathbb{E}(Q) > 0. \quad (2.84)$$

The linearized operator  $i\mathcal{L}^\omega$  has the generalized kernel

$$i\mathcal{L}^\omega iQ_\omega = 0, \quad i\mathcal{L}^\omega Q'_\omega = -iQ_\omega, \quad (2.85)$$

where  $Q'_\omega = S_\omega \omega \Psi'[\omega] = Q' + O(\omega^{-1/4})$  in  $H^1$ , and a real eigenvalue  $\alpha_\omega = \alpha + O(\omega^{-1/4}) \in (\frac{9}{10}\alpha, \frac{11}{10}\alpha)$  with the eigenfunctions

$$i\mathcal{L}^\omega g_\pm^\omega = \pm \alpha_\omega g_\pm^\omega, \quad g_\pm^\omega = g_1^\omega \pm i g_2^\omega = g_\pm + O(\omega^{-1/4}) \text{ in } H^1, \quad (2.86)$$



satisfying (2.68). There are constants  $\delta_C, \delta_D \in (0, 1)$  such that for any  $\omega \geq \omega_*$ ,

$$\mathcal{C}_\omega : (\theta, b_+, b_-, \zeta) \mapsto e^{i\theta}(Q_\omega + b_+g_+^\omega + b_-g_-^\omega + \zeta) \quad (2.87)$$

is a diffeomorphism from the set

$$\mathcal{U}_\omega := \{(\theta, b_+, b_-, \zeta) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^2 \times \mathbb{Z}_\omega \mid |b_+|^2 + |b_-|^2 + \|\zeta\|_{H^1}^2 < \delta_C^2\} \quad (2.88)$$

into  $H_r^1$ , whose image contains the following neighborhood of  $\mathcal{Q}_\omega$

$$\mathcal{N}_\omega := \{\varphi \in H_r^1(\mathbb{R}^3) \mid d_{0,\omega}(\varphi) < \delta_D\} \subset \mathcal{C}_\omega(\mathcal{U}_\omega). \quad (2.89)$$

From  $\varphi := \mathcal{C}_\omega(\theta, b_+, b_-, \zeta) \in \mathcal{N}_\omega$  we can recover

$$\begin{aligned} \theta &= \arg(\varphi|Q'_\omega), \quad (b_+, b_-, \zeta) = (P_+^\omega, P_-^\omega, P_c^\omega)(e^{-i\theta}\varphi - Q_\omega), \\ b_1 &:= b_+ + b_-, \quad b_2 := b_+ - b_-, \end{aligned} \quad (2.90)$$

where  $\langle \varphi|Q'_\omega \rangle > 0$ . For later use, they are denoted by

$$\mathcal{B}_*^\omega(\varphi) := b_* \quad (* = +, -, 1, 2), \quad \mathcal{Z}^\omega(\varphi) := \zeta \quad (2.91)$$

The linearized energy norm  $\|v\|_\omega$  is equivalent to the  $H^1$  norm uniformly for  $\omega \geq \omega_*$ .

### 3. DYNAMICS AROUND THE FIRST EXCITED STATE

**3.1. Expansion around the excited state.** For any solution  $u$  of (1.1) and any  $\omega > 0$ , consider the parabolic rescaling which preserves the equation and  $\dot{H}^{1/2}(\mathbb{R}^3)$

$$u_\omega(t, x) := S_\omega u(t/\omega) = \omega^{-1/2}u(\omega^{-1}t, \omega^{-1/2}x). \quad (3.1)$$

Then (1.1) is rescaled to

$$(i\partial_t - \Delta + V^\omega)u_\omega = |u_\omega|^2 u_\omega. \quad (3.2)$$

We say that a solution  $u_\omega$  of (3.2) either *blows up*, *scatters to  $\Phi$* , or *is trapped by  $\Psi$* , if it happens for the unscaled solution  $u(t) := S_\omega^{-1}u_\omega(\omega t)$  (cf. Section 1.2).

Suppose that  $u_\omega$  is close to the orbit of the excited state  $\mathcal{Q}_\omega$  at some  $t$ . More precisely, assume  $u_\omega(t) \in \mathcal{N}_\omega$  or  $d_{0,\omega}(u_\omega(t)) < \delta_D$  for some  $\omega \geq \omega_*$  at some  $t \in \mathbb{R}$ . Here we could restrict  $\omega$  by specifying the mass  $\mathbb{M}(u_\omega) = \mathbb{M}(Q_\omega)$  or equivalently  $\mathbb{M}(u) = \mathbb{M}(\Psi[\omega])$  as in [7], but it is more convenient to keep the freedom of  $\omega$  in constructing the center-stable manifold (see Section 7).

Expanding the solution  $u_\omega$  of (3.2) in the form

$$u_\omega(t, x) = e^{i\theta(t)}(Q_\omega(x) + v(t, x)) \quad (3.3)$$

with  $\theta(t) \in \mathbb{R}/2\pi\mathbb{Z}$  and  $v(t) \in H^1(\mathbb{R}^3)$  yields an equation for  $v$

$$i\dot{v} = -\mathcal{L}^\omega v + (\dot{\theta} + 1)(Q_\omega + v) + N^\omega(v), \quad (3.4)$$

where  $N^\omega : H^1 \rightarrow H^{-1}$  is the Fréchet derivative of  $C^\omega$  given by

$$N^\omega(v) := 2Q_\omega|v|^2 + Q_\omega v^2 + |v|^2 v. \quad (3.5)$$

In the real value, the equation is written as

$$\begin{cases} \dot{v}_1 = -L_-^\omega v_2 + (\dot{\theta} + 1)v_2 + 2Q_\omega v_1 v_2 + |v|^2 v_2, \\ \dot{v}_2 = L_+^\omega v_1 - (\dot{\theta} + 1)(Q_\omega + v_1) - Q_\omega(3v_1^2 + v_2^2) - |v|^2 v_1. \end{cases} \quad (3.6)$$

**3.2. Orthogonality and equations.** In order to exploit the coercivity of  $\mathcal{L}^\omega$ , we choose  $\theta(t)$  by the local coordinate  $\mathcal{C}_\omega$ , see (2.90), or the orthogonality

$$0 = \langle iQ'_\omega | v \rangle = \langle iQ'_\omega | e^{-i\theta} u_\omega \rangle, \quad \|v\|_{H^1} \sim d_{0,\omega}(u_\omega). \quad (3.7)$$

Differentiating the above orthogonality condition in  $t$  yields

$$0 = \partial_t \langle i v | Q'_\omega \rangle = \langle v | Q_\omega \rangle + (\dot{\theta} + 1) \langle Q_\omega + v | Q'_\omega \rangle + \langle N^\omega(v) | Q'_\omega \rangle, \quad (3.8)$$

which can be rewritten as an equation for  $\theta(t)$

$$\dot{\theta} + 1 = m^\omega(v), \quad (3.9)$$

where  $m^\omega(v)$  is defined and  $C^1$  for small  $v \in H_r^1$  by the equation

$$0 = \langle Q_\omega + v | Q'_\omega \rangle m^\omega(v) + \langle v | Q_\omega \rangle + \langle N^\omega(v) | Q'_\omega \rangle. \quad (3.10)$$

Since  $\langle u_\omega | Q'_\omega \rangle > 0$  as long as  $u_\omega \in \mathcal{N}_\omega$  (cf. (2.90)),  $m^\omega(v)$  is well-defined, satisfying

$$|m^\omega(v)| \lesssim \|P_\mathfrak{c}^\omega v\|_2 + \|v\|_{H^1}^2, \quad (3.11)$$

since  $\langle g_\pm^\omega | Q_\omega \rangle = 0$ . Plugging it into (3.4) yields an autonomous equation of  $v$

$$\dot{v} = i\mathcal{L}^\omega v - m^\omega(v)iQ_\omega + \mathcal{N}^\omega(v), \quad \mathcal{N}^\omega(v) := -i(m^\omega(v)v + N^\omega(v)). \quad (3.12)$$

It can be rewritten in the local coordinate of  $\mathcal{C}_\omega$ . Denoting

$$\mathcal{N}_*^\omega(v) := P_*^\omega \mathcal{N}^\omega(v) \quad (3.13)$$

for  $* = \pm, 1, 2, \mathfrak{c}$ , we obtain the following equations for each  $b_* = P_*^\omega v$

$$\dot{b}_\pm = \pm \alpha_\omega b_\pm + \mathcal{N}_\pm^\omega(v), \quad \begin{cases} \dot{b}_1 = \alpha_\omega b_2 + \mathcal{N}_1^\omega(v), \\ \dot{b}_2 = \alpha_\omega b_1 + \mathcal{N}_2^\omega(v), \end{cases} \quad (3.14)$$

as well as for  $\zeta = P_\mathfrak{c}^\omega v$

$$\dot{\zeta} = i\mathcal{L}^\omega \zeta - m^\omega(v)iQ_\omega + \mathcal{N}_\mathfrak{c}^\omega(v). \quad (3.15)$$

**3.3. Energy distance function.** In view of the expansion (2.80), the linearized energy norm  $\|v\|_\omega$  is better suited than  $d_{0,\omega}$  to measure the distance from  $\mathcal{S}_1$  to solutions  $u$ . In order to avoid the regularity loss from the higher order term  $C^\omega$ , we can either include it into the distance, or mollify the distance in time. Here we take the latter option for a simpler (convex) dynamics of the (square) distance.

Fix a radial decreasing function  $\chi \in C_0^\infty(\mathbb{R})$  satisfying

$$\chi(t) = \begin{cases} 1 & |t| \leq 1, \\ 0 & |t| \geq 2. \end{cases} \quad (3.16)$$

For any  $\varphi \in \mathcal{N}_\omega$ , let  $u_\omega$  be the solution of (3.2) with  $u_\omega(0) = \varphi$ . Decomposing  $u_\omega = e^{i\theta}(Q_\omega + v)$  as above, a local distance function  $d_{1,\omega}$  is defined at  $\varphi$  by

$$d_{1,\omega}(\varphi)^2 := \int_{\mathbb{R}} \chi(-t) \|v(t)\|_\omega^2 dt. \quad (3.17)$$

The local wellposedness for (3.12) in  $v \in H^1(\mathbb{R}^3)$ , which is uniform in  $\omega$ , yields

**Lemma 3.1.** *There are constants  $\delta_E \in (0, \delta_D/2]$  and  $C \in [1, \infty)$  such that for any  $\omega \geq \omega_*$  and any  $\varphi \in H_r^1(\mathbb{R}^3)$  with  $d_{0,\omega}(\varphi) \leq 2\delta_E$ , the solution  $u_\omega$  of (3.2) with the initial condition  $u_\omega(0) = \varphi$  exists at least for  $|t| \leq 2$ , satisfying*

$$|t| \leq 2 \implies C^{-1}d_{0,\omega}(u_\omega(0)) \leq d_{0,\omega}(u_\omega(t)) \leq Cd_{0,\omega}(u_\omega(0)). \quad (3.18)$$

Hence  $d_{1,\omega}$  is well-defined for  $d_{0,\omega}(\varphi) \leq 2\delta_E$  and uniformly equivalent to  $d_{0,\omega}$ . Then a global distance function  $d_\omega : H_r^1 \rightarrow [0, \infty)$  is defined by

$$d_\omega(\varphi) := \chi(d_{0,\omega}(\varphi)/\delta_E)d_{1,\omega}(\varphi) + (1 - \chi(d_{0,\omega}(\varphi)/\delta_E))d_{0,\omega}(\varphi). \quad (3.19)$$

$d_\omega : H_r^1 \rightarrow [0, \infty)$  satisfies  $d_\omega(\varphi) \sim d_{0,\omega}(\varphi)$  uniformly and

$$d_{0,\omega}(\varphi) \leq \delta_E \implies d_\omega(\varphi) = d_{1,\omega}(\varphi). \quad (3.20)$$

**3.4. Instability and ejection.** The crucial property of the dynamics around  $\mathcal{S}_1$  is that the rescaled solution  $u_\omega$  can get away from  $Q_\omega$  only by growing instability. More precisely, we have

**Lemma 3.2.** *There are constants  $c_X \in (0, 1)$  and  $\delta_I \in (0, \delta_E]$  such that for any  $\omega \geq \omega_*$  and any  $\varphi \in H_r^1(\mathbb{R}^3)$  we have uniformly*

$$\left. \begin{aligned} \mathbb{A}^\omega(\varphi) - \mathbb{A}^\omega(Q_\omega) &\leq c_X d_\omega(\varphi)^2 \\ \text{and } d_\omega(\varphi) &\leq \delta_I \end{aligned} \right\} \implies d_\omega(\varphi) = d_{1,\omega}(\varphi) \sim |\mathcal{B}_1^\omega(\varphi)|. \quad (3.21)$$

*Proof.* Since  $d_\omega \sim d_{0,\omega}$  and (3.20), choosing  $\delta_I$  small ensures that  $d_\omega(\varphi) = d_{1,\omega}(\varphi)$  for  $d_\omega(\varphi) \leq \delta_I$ . Then by the definition of  $d_{1,\omega}$  and equivalence of distance functions,

$$\|v\|_\omega^2 = \mathbb{A}^\omega(\varphi) - \mathbb{A}^\omega(Q_\omega) + b_1^2 - C^\omega(v) \lesssim (c_X + \delta_I)\|v\|_\omega^2 + b_1^2, \quad (3.22)$$

where  $v \in \mathcal{V}_\omega$  is determined from  $\varphi$  by (2.76) as before. Choosing  $c_X$  and  $\delta_I$  small enough, we obtain  $d_\omega(\varphi)^2 \sim \|v\|_\omega^2 \lesssim b_1^2$ .  $\square$

Next we investigate the evolution of  $d_\omega$ . For any solution  $u_\omega$  of (3.2) close to  $Q_\omega$ ,

$$d_{1,\omega}(u_\omega)^2 = \chi * \|v\|_\omega^2, \quad (3.23)$$

where  $*$  denotes the convolution in  $t$ , and  $u_\omega = e^{i\theta}(Q_\omega + v)$  with the orthogonality  $v \in \mathcal{V}_\omega$  as before. Then using the equation (3.14) for  $b_j$ , (2.80) and conservation of  $\mathbb{A}^\omega(u_\omega)$ , we derive

$$\begin{aligned} \partial_t d_{1,\omega}(u_\omega)^2 &= \chi * 2\alpha_\omega[b_1 b_2 + O(\|v\|_{H^1}^3)] + \chi' * O(\|v\|_{H^1}^3), \\ \partial_t^2 d_{1,\omega}(u_\omega)^2 &= \chi * 2\alpha_\omega^2[b_1^2 + b_2^2] + \sum_{j=0}^2 \chi^{(j)} * O(\|v\|_{H^1}^3). \end{aligned} \quad (3.24)$$

Note that we can not differentiate the cubic terms, that is the reason for the mollifier. If  $u_\omega(t)$  is in the instability dominance (3.21), then

$$\begin{aligned} \partial_t d_\omega(u_\omega)^2 &= \chi * 2\alpha_\omega[b_1 b_2] + O(b_1^3), \\ \partial_t^2 d_\omega(u_\omega)^2 &= \chi * 2\alpha_\omega^2[b_1^2 + b_2^2] + O(b_1^3) \sim b_1^2, \end{aligned} \quad (3.25)$$

using the equivalence (3.18) as well, and assuming if necessary that  $d_\omega(u_\omega)$  is even smaller. The last equation implies the convexity of  $d_{1,\omega}^2$ , from which we deduce that if  $u_\omega$  satisfies at some time  $t = t_0$ ,

$$d_\omega(u_\omega) < \delta_X, \quad \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega) \leq c_X d_\omega(u_\omega)^2, \quad \partial_t d_\omega(u_\omega) \geq 0, \quad (3.26)$$

for some small  $\delta_X \leq \delta_I$ , then  $d_\omega(u_\omega)$  will keep growing for  $t \geq t_0$  until the first condition is violated.  $d_\omega(u_\omega) \sim |b_1|$  implies that  $\sigma := \text{sign } b_1(t) \in \{\pm 1\}$  is fixed during that time.

Since  $0 \leq \partial_t d_\omega(u_\omega) = \partial_t d_{1,\omega}(u_\omega)$ , the first equation in (3.25) implies that  $b_1 b_2(t_1) \gtrsim -|b_1(t_1)|^3$  at some  $t_1 \in (t_0 - 2, t_0 + 2)$ , then (taking  $\delta_X$  small),  $\sigma b_+(t_1) \geq |b_1(t_1)|/3$  and  $\sigma b_-(t_1) \geq 0$ . Because  $\partial_t(b_1 b_2) \geq 0$  and  $b_1 \sim b_1(t_0)$  for  $|t - t_0| < 2$ , we may choose  $t_1 > t_0$ .

Let  $R := |b_1(t_1)|$  and suppose that on some interval  $[t_1, t_2]$  we have (3.26) and

$$|b_1| \leq 2Re^{\alpha_\omega(t-t_1)}. \quad (3.27)$$

Then the equations of  $b_\pm$  in (3.14) imply

$$|b_\pm - e^{\pm\alpha_\omega(t-t_1)} b_\pm(t_1)| \lesssim (Re^{\alpha_\omega(t-t_1)})^2 \lesssim \delta_X Re^{\alpha_\omega(t-t_1)}, \quad (3.28)$$

and so, taking  $\delta_X$  smaller if necessary,

$$|b_1| = \sigma(b_+ + b_-) \begin{cases} \leq Re^{\alpha_\omega(t-t_1)}(1 + C\delta_X) < 2Re^{\alpha_\omega(t-t_1)}, \\ \geq Re^{\alpha_\omega(t-t_1)}(1/3 - C\delta_X) > Re^{\alpha_\omega(t-t_1)}/4. \end{cases} \quad (3.29)$$

Therefore  $t_2$  can be increased until  $d_\omega(u_\omega)$  reaches  $\delta_X$  at some  $t = t_X > t_0$ , and for  $t_0 \leq t \leq t_X$ , we have

$$d_\omega(u_\omega) \sim \sigma b_1(t) \sim \sigma b_1(t_0)e^{\alpha_\omega(t-t_0)}, \quad (3.30)$$

for a time-independent sign  $\sigma \in \{\pm 1\}$ , while the equations of  $b_\pm$  imply

$$b_\pm(t) = e^{\pm\alpha_\omega(t-t_0)} b_\pm(t_0) + O(b_1^2). \quad (3.31)$$

To estimate  $\zeta$ , consider the energy projected onto the eigenmodes

$$\mathbb{A}^\omega(bg^\omega) = \frac{1}{2} [-b_1^2 + b_2^2] - C^\omega(bg^\omega), \quad bg^\omega := b_1 g_1^\omega + ib_2 g_2^\omega. \quad (3.32)$$

Using the equations of  $b$ , we have, for  $t_0 < t < t_X$ ,

$$\begin{aligned} \partial_t \mathbb{A}^\omega(bg^\omega) &= -b_1(\alpha_\omega b_2 + \mathcal{N}_1^\omega(v)) + b_2(\alpha_\omega b_1 + \mathcal{N}_2^\omega(v)) - \langle N^\omega(bg^\omega) | \dot{bg}^\omega \rangle \\ &= \alpha_\omega m^\omega(v) [b_1 \langle v_2 | g_2^\omega \rangle + b_2 \langle v_1 | g_1^\omega \rangle] = O(\|\zeta\|_\omega \|v\|_\omega^2 + \|v\|_\omega^4), \end{aligned} \quad (3.33)$$

where (3.11) is used. On the other hand

$$\begin{aligned} \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega) - \mathbb{A}^\omega(bg^\omega) &= \frac{1}{2} \langle \mathcal{L}^\omega \zeta | \zeta \rangle - C^\omega(v) + C^\omega(bg^\omega) \\ &= \|\zeta\|_\omega^2 + O(\|\zeta\|_\omega \|v\|_\omega^2). \end{aligned} \quad (3.34)$$

Hence integrating its time derivative in  $[t_0, t_X]$  leads to

$$\|\zeta\|_{L_t^\infty H^1}^2 \lesssim \|\zeta(t_0)\|_{H^1}^2 + |b_1(t_0)|^4 + \|b_1\|_{L_t^4}^4, \quad (3.35)$$

so, using the exponential growth of  $b_1$ ,

$$\|\zeta(t)\|_{H^1} \lesssim \|\zeta(t_0)\|_{H^1} + |b_1(t)|^2. \quad (3.36)$$

Near  $t = t_X$ , we can also determine the sign and size of

$$\begin{aligned}\mathbb{K}_2^\omega(u_\omega) &= \langle (\mathbb{K}_2^\omega)'(Q_\omega) | v \rangle + O(\|v\|_{H^1}^2) \\ &= b_1 \langle (\mathbb{K}_2^\omega)'(Q_\omega) | g_1^\omega \rangle + O(\|\zeta_1\|_\omega + \|v\|_\omega^2).\end{aligned}\tag{3.37}$$

(2.68) implies that for some constant  $C_K > 0$

$$\sigma \mathbb{K}_2^\omega(u_\omega) + C_K \|\zeta(t_0)\|_{H^1} \sim \sigma b_1 = |b_1| \tag{3.38}$$

on  $t_0 \leq t \leq t_X$ . Thus we have obtained the following.

**Lemma 3.3** (Ejection Lemma). *There are constants  $C_K \in (0, \infty)$  and  $\delta_X \in (0, \delta_I]$  such that for any  $\omega \geq \omega_*$  and any solution  $u_\omega$  of (3.2) satisfying the three conditions in (3.26) at some  $t = t_0 \in \mathbb{R}$ , there exists  $t_X \in (t_0, \infty)$  such that  $d_\omega(u_\omega(t_X)) = \delta_X$ , and for  $t_0 < t < t_X$ ,  $d_\omega(u_\omega(t))$  is strictly increasing,*

$$\begin{aligned}d_\omega(u_\omega(t)) &\sim \sigma \mathcal{B}_1^\omega(u_\omega(t)) \sim \sigma \mathcal{B}_+^\omega(u_\omega(t)) \sim \sigma \mathcal{B}_1^\omega(u_\omega(t_0)) e^{\alpha_\omega(t-t_0)} \\ &\sim \sigma \mathbb{K}_2^\omega(u_\omega(t)) + C_K \|\mathcal{Z}^\omega(u_\omega(t_0))\|_{H^1},\end{aligned}\tag{3.39}$$

and  $\mathcal{B}_\pm^\omega(u_\omega(t)) = e^{\pm \alpha_\omega(t-t_0)} \mathcal{B}_\pm^\omega(u_\omega(t_0)) + O(d_\omega(u_\omega(t))^2)$ , for some  $\sigma \in \{\pm 1\}$  independent of  $t \in (t_0, t_X)$ .

*Remark 3.4.* In the previous papers such as [7], the sign was opposite between the unstable mode and the virial functional. In this paper, the sign of the eigenmode is chosen to match that of virial. In other words, the sign of eigenmode is flipped from [7], by the choice (normalization) of  $g_\pm^\omega$ .

Note that by the time inversion symmetry, we can and will apply the above lemma backward in time as well. As an immediate consequence, we can describe the behavior of solutions which are not ejected. In contrast to the ejected solutions, they are monotonically and exponentially attracted by a small neighborhood of  $Q_\omega$ .

**Lemma 3.5** (Trapping Lemma). *Let  $\omega \geq \omega_*$  and  $u_\omega$  be a solution of (3.2) on some interval  $[t_0, \infty)$  satisfying*

$$\sup_{t_0 < t < \infty} d_\omega(u_\omega(t)) < \delta_X. \tag{3.40}$$

*Then there exists  $t_1 \in [t_0, \infty]$  such that  $d_\omega(u_\omega(t))$  is strictly decreasing on  $[t_0, t_1]$  and*

$$\begin{cases} t_0 \leq t < t_1 \implies d_\omega(u_\omega(t)) \sim e^{-\alpha_\omega(t-t_0)} d_\omega(u_\omega(t_0)), \\ t_1 < t < \infty \implies c_X d_\omega(u_\omega(t))^2 < \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega). \end{cases} \tag{3.41}$$

*We have  $t_1 = \infty$  if and only if  $u_\omega(t)$  converges to  $e^{i(a-t)} Q_\omega$  strongly in  $H^1(\mathbb{R}^3)$  as  $t \rightarrow \infty$  for some  $a \in \mathbb{R}$ . Moreover, in that case we have*

$$\|u_\omega(t) - e^{i(a-t)} Q_\omega\|_{H^1} \sim e^{-\alpha_\omega(t-t_0)} d_\omega(u_\omega(t_0)). \tag{3.42}$$

*Proof.* Abbreviate  $d(t) := d_\omega(u_\omega(t))$ . The ejection lemma 3.3 implies that if (3.26) holds at any  $t \in [t_0, \infty)$  then  $d(t)$  grows at least to  $\delta_X$ , violating the first condition of (3.40). Since  $d(t) < \delta_X$  for all  $t \geq t_0$ , the latter two conditions of (3.26) should never hold, in other words

$$\partial_t d(t) < 0 \quad \text{or} \quad c_X d(t)^2 < \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega). \tag{3.43}$$

Then  $t_1 := \inf\{t \geq t_0 \mid c_X d(t)^2 < \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega)\}$  satisfies the desired properties. The exponential decay follows from Lemma 3.3 applied backward in time

$$d(t_0) \sim e^{\alpha_\omega(t_0-t)} d(t) \quad (3.44)$$

for  $t_0 < t < t_1$ . If  $t_1 = \infty$  then  $d(t) \sim e^{-\alpha_\omega(t-t_0)} d(t_0) \rightarrow 0$  as  $t \rightarrow \infty$ , hence exponential convergence to the set  $\mathcal{Q}_\omega$ . Conversely, if  $u_\omega$  is strongly convergent, then  $\mathbb{A}^\omega(u_\omega) = \mathbb{A}^\omega(Q_\omega)$ , which forces  $t_1 = \infty$ . Then the modulation equation (3.9) yields  $|\dot{\theta} + 1| \lesssim \|v\|_{H^1} \lesssim e^{-\alpha_\omega(t-t_0)} \delta$ , and by integration in  $t$ , the same bound for  $|\theta - (a - t)|$  for some  $a \in \mathbb{R}$ .  $\square$

Under an energy constraint  $\mathbb{A}^\omega(u_\omega) < \mathbb{A}^\omega(Q_\omega) + c_X \delta^2$  for some  $\delta \in (0, \delta_X)$ , every solution  $u_\omega$  satisfying (3.40) comes closer to  $\mathcal{Q}_\omega$ , namely

$$t_1 < t < \infty \implies d_\omega(u_\omega(t)) < \delta < \delta_X. \quad (3.45)$$

This distance gap between the ejection and the trapping is a key property of the instability dynamics.

#### 4. STATIC ANALYSIS AWAY FROM THE FIRST EXCITED STATE

When the solution is away from the first excited states, the above linearization is useless. Instead, we rely on energy-type, variational and topological arguments.

**4.1. Variational estimate away from the solitons.** We have sign-definiteness of the virial  $\mathbb{K}_2^\omega$  away from the solitons.

**Lemma 4.1.** *There exist continuous, positive and increasing functions  $\varepsilon_V$  and  $\kappa_V$  from  $(0, \infty)$  to  $(0, 1)$  with the following property. Let  $\omega \geq \omega_*$  and  $\varphi \in H_r^1(\mathbb{R}^3)$  satisfy the following three conditions:*

$$d_\omega(\varphi) \geq \delta, \quad \mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + \varepsilon_V(\delta)^2, \quad \mathbb{M} + \omega \mathbb{H}^0(\varphi) > C_M. \quad (4.1)$$

Then we have one of the following (a)-(b).

- (a)  $|\mathbb{K}_2^\omega(\varphi)| \geq \kappa_V(\delta)$ .
- (b)  $\mathbb{H}^0(\varphi) \leq C_D \mathbb{K}_2^\omega(\varphi) \leq C_D^2 \mathbb{H}^0(\varphi)$  and  $\mathbb{E}^\omega(\varphi) < \delta_U/2$ , where

$$\delta_U := \inf_{\omega \geq \omega_*} \mathbb{E}^\omega(Q_\omega) > 0. \quad (4.2)$$

Note that the first and the third conditions in (4.1) are to avoid the sign change of  $\mathbb{K}_2^\omega$  respectively around the first excited state and around the ground state, but we can not avoid the vanishing at 0, namely  $\|\nabla \varphi\|_2 \rightarrow 0$  as  $\omega \rightarrow \infty$ , corresponding to the case (b).  $\delta_U > 0$  is ensured by (2.84).

*Proof.* We argue by contradiction. Let  $(\varphi, \omega) = (\varphi_n, \omega_n)$  be a sequence in  $H_r^1 \times [\omega_*, \infty)$  such that as  $n \rightarrow \infty$ ,

$$d_\omega(\varphi) \geq \delta, \quad \mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + o(1), \quad (\mathbb{M} + \omega \mathbb{H}^0)(\varphi) > C_M, \quad \mathbb{K}_2^\omega(\varphi) \rightarrow 0, \quad (4.3)$$

and that  $\varphi$  does not satisfy (b). Combining the above with (2.7), (2.83) and (2.84) yields

$$(\mathbb{M} + \mathbb{H}^0/3)(\varphi) < 2\mathbb{A}(Q), \quad (4.4)$$

for large  $n$ , so that we can extract a subsequence such that  $\omega$  converges to some  $\varpi \in [\omega_*, \infty]$  and that  $\varphi$  converges to some  $\varphi_\infty$  weakly in  $H_r^1$  and strongly in  $L^4$ . The convergence implies

$$\mathbb{J}^\varpi(\varphi_\infty) \leq \mathbb{J}^\varpi(Q_\varpi), \quad \mathbb{K}_2^\varpi(\varphi_\infty) \leq 0. \quad (4.5)$$

Apply Lemma 2.1 to  $\varphi$ . Lemma 2.3 (1) with (4.4) precludes the case (2.9). Since (2.10) with  $\mathbb{K}_2^\omega(\varphi) \rightarrow 0$  would lead to (b) for large  $n$ , we deduce that  $\varphi$  is in the case (2.11), so is the weak limit  $\varphi_\infty$ . Then Lemma 2.3 implies that  $\varphi_\infty = e^{i\theta} Q_\varpi$  for some  $\theta \in \mathbb{R}$ , and so  $\mathbb{A}^\omega(\varphi) \rightarrow \mathbb{A}^\varpi(\varphi_\infty)$ , which implies that the convergence  $\varphi \rightarrow e^{i\theta} Q_\varpi$  is strong in  $H^1$ , contradicting  $d_\omega(\varphi) \geq \delta$ .  $\square$

**4.2. Sign functional.** Combining the above Lemmas 3.3 and 4.1, we see that the sign  $\sigma$  in (3.39) can be given by a functional on two separate open sets in  $H_r^1$  away from  $Q_\omega$ . More precisely, we have

**Lemma 4.2.** *There exist constants  $\delta_V \in (0, \delta_X/2)$  and  $\varepsilon_S \in (0, \varepsilon_V(\delta_V))$  such that for each  $\omega \geq \omega_*$  there exists a unique continuous functional*

$$\mathfrak{S}_\omega : \{\varphi \in H_r^1(\mathbb{R}^3) \mid \mathbb{A}^\omega(\varphi) - \mathbb{A}^\omega(Q_\omega) < \min(\varepsilon_S^2, c_X d_\omega(\varphi)^2)\} =: \check{\mathcal{H}}_\omega \rightarrow \{\pm 1\} \quad (4.6)$$

satisfying (i)–(iii) below. For any  $\varphi \in \check{\mathcal{H}}_\omega$  and  $\theta \in \mathbb{R}$ ,

- (i) If  $C_S \mathbb{M} \mathbb{H}^0(\varphi) \leq 1$  then  $\mathfrak{S}_\omega(\varphi) = +1$ .
- (ii) If  $d_\omega(\varphi) < 2\delta_V$  then  $\mathfrak{S}_\omega(\varphi) = \text{sign } \mathcal{B}_1^\omega(\varphi)$ .
- (iii) If  $\mathbb{A}^\omega(\varphi) - \mathbb{A}^\omega(Q_\omega) < \varepsilon_V(d_\omega(\varphi))^2$  and  $(\mathbb{M} + \omega \mathbb{H}^0)(\varphi) > C_M$ , then  $\mathfrak{S}_\omega(\varphi) = \text{sign } \mathbb{K}_2^\omega(\varphi)$ .

Moreover, if  $\varphi \in H_r^1(\mathbb{R}^3)$  satisfies  $S_{\omega^\perp} \varphi \in \check{\mathcal{H}}_{\omega^\perp}$  for some  $\omega^\perp, \omega^\perp \geq \omega_*$ , then

$$\mathfrak{S}_{\omega^\perp}(S_{\omega^\perp} \varphi) = \mathfrak{S}_{\omega^\perp}(S_{\omega^\perp} \varphi). \quad (4.7)$$

If  $\mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + \varepsilon_S^2$  and  $d_\omega(\varphi) > \delta_V$ , then  $\mathbb{A}^\omega(\varphi) - \mathbb{A}^\omega(Q_\omega) < \varepsilon_V(\delta_V)^2 < \varepsilon_V(d_\omega(\varphi))^2$ . If  $(\mathbb{M} + \omega \mathbb{H}^0)(\varphi) \leq C_M$  then  $C_S \mathbb{M} \mathbb{H}^0(\varphi) \leq 1$  by Lemma 2.3(2). Hence (i)–(iii) determine the value of  $\mathfrak{S}_\omega$  on  $\check{\mathcal{H}}_\omega$ , which is independent of the choice of  $\delta_V$  and  $\varepsilon_S$  (because (i) and (iii) are independent). The continuity of  $\mathfrak{S}_\omega$  simply means that it is constant on each connected component of  $\check{\mathcal{H}}_\omega$ . The last sentence of lemma allows us to define a functional independent of  $\omega$

$$\begin{aligned} \mathfrak{S} : \check{\mathcal{H}} &\rightarrow \{\pm 1\}, \quad \mathfrak{S}(\varphi) := \mathfrak{S}_\omega(S_\omega \varphi), \\ \check{\mathcal{H}} &:= \{\varphi \in H_r^1(\mathbb{R}^3) \mid \exists \omega > \omega_*, S_\omega \varphi \in \check{\mathcal{H}}_\omega\}. \end{aligned} \quad (4.8)$$

As a sufficient condition for  $\varphi \in \check{\mathcal{H}}$ , using  $\inf_{\omega > 0} \mathbb{A}^\omega(S_\omega \varphi) = 2\sqrt{\mathbb{E}(\varphi)\mathbb{M}(\varphi)}$  and

$$d_\omega(S_\omega \varphi) \sim \text{dist}_{H^1}(S_\omega \varphi, Q_\omega) \gtrsim \text{dist}_{\dot{H}^{1/2}}(\varphi, \{e^{i\theta} \Psi[\omega]\}_\theta), \quad (4.9)$$

we see that there exist  $0 < \mu, \varepsilon, c \ll 1$  such that  $\varphi \in H_r^1(\mathbb{R}^3)$  belongs to  $\check{\mathcal{H}}$  if

$$\mathbb{M}(\varphi) < \mu \quad \text{and} \quad \mathbb{E}(\varphi)\mathbb{M}(\varphi) < \mathbb{E}^0(Q)\mathbb{M}(Q) + \min(\varepsilon^2, c \text{dist}_{\dot{H}^{1/2}}(\varphi, \mathcal{S}_1)). \quad (4.10)$$

Notice that  $\sigma$  in the ejection lemma 3.3 is *not necessarily* equal to  $\mathfrak{S}_\omega(u_\omega(t_X))$ , but it is so if the solution is well accelerated at the ejection time  $t_X$ , that is the case if  $d_\omega(u_\omega(t_0)) \ll \delta_X$ . In any case, the sign functional  $\mathfrak{S}_\omega$  will give the correct prediction of dynamics after the ejection. It is also worth noting

**Lemma 4.3.**  $\mathfrak{S}_\omega^{-1}(\{+1\})$  is uniformly bounded in  $H^1$  for  $\omega \geq \omega_*$ .

*Proof.* It is obvious in the case (ii) of Lemma 4.2, because  $Q_\omega$  is bounded. In the case (iii), the uniform bound follows from (2.7) and  $\mathbb{K}_2^\omega(\varphi) \geq 0$ . In the case (i), using (2.36) and that  $Q$  attains the best constant in Gagliardo-Nirenberg  $\mathbb{G} \lesssim (\mathbb{M}\mathbb{H}^0)^{1/2}\mathbb{H}^0$ , we obtain

$$\mathbb{G}(\varphi) \leq \frac{\mathbb{G}(Q)}{\mathbb{M}(Q)^{1/2}\mathbb{H}^0(Q)^{3/2}} C_S^{-1/2} \mathbb{H}^0(\varphi) \leq \frac{\mathbb{G}(Q)}{\mathbb{H}^0(Q)} \mathbb{H}^0(\varphi) = \frac{2}{3} \mathbb{H}^0(\varphi), \quad (4.11)$$

where we also used the Pohozaev identity, cf. (2.3). Using (2.83) as well, we obtain

$$(i) \implies \mathbb{A}^\omega(\varphi) = (\mathbb{H}^0 + \mathbb{M} + \lceil V^\omega \rceil - \mathbb{G})(\varphi) \geq \frac{2}{9} \mathbb{H}^0(\varphi) + \frac{9}{10} \mathbb{M}(\varphi). \quad (4.12)$$

Since the cases (i)-(iii) exhaust the region  $\check{\mathcal{H}}_\omega$  as seen above, we conclude that  $\mathfrak{S}_\omega^{-1}(\{+1\})$  is uniformly bounded.  $\square$

*Proof of Lemma 4.2.* Fix  $0 < \delta_V \ll \delta_X$  and  $\varepsilon_S \in (0, \varepsilon_V(\delta_V))$ . To show that  $\mathfrak{S}_\omega$  is uniquely, continuously and well defined by (i)-(iii), it suffices to show that (i), (ii) and (iii) do not contradict in the intersections.

There is no intersection of (i) and (ii) because of (2.36) and (2.84), if  $\delta_V > 0$  is small enough. Choosing  $\varepsilon_S$  small enough and using (4.12),  $\varphi \in \check{\mathcal{H}}_\omega$  and (2.84), we have

$$(i) \implies \mathbb{M}(\varphi) < \frac{10}{9} (\mathbb{A}^\omega(Q_\omega) + \varepsilon_S^2) < 2\mathbb{A}(Q). \quad (4.13)$$

Hence in the intersection of (i) and (iii), Lemma 2.3 (1)-(2) precludes (2.9) and (2.11), then (2.10) implies  $\mathbb{K}_2^\omega(\varphi) > 0$ .

For the intersection of (ii) and (iii), let  $\varphi \in \check{\mathcal{H}}_\omega$  satisfy  $\mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + \varepsilon_V(d_\omega(\varphi))$  and  $d_\omega(\varphi) < 2\delta_V$ . Let  $u_\omega$  be the solution of (3.2) with  $u_\omega(0) = \varphi$ . Since  $\varphi$  satisfies (3.21), the ejection lemma 3.3 is applied to  $u_\omega$ , either forward or backward in time from  $t = 0$ . In both cases, there exists  $t_X \in \mathbb{R}$  such that  $d_\omega(u_\omega) \in (d_\omega(\varphi), \delta_X)$  is monotone between  $t = 0$  and  $t_X$ , with  $d_\omega(u_\omega(t_X)) = \delta_X$ . Since  $d_\omega(\varphi) < 2\delta_V \ll \delta_X$ , (3.39) implies that  $\sigma = \text{sign } \mathbb{K}_2^\omega(u_\omega(t_X)) = \text{sign } b_1(0)$ . Since  $\mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega) < \varepsilon_V(d_\omega(\varphi))^2 \leq \varepsilon_V(u_\omega)^2$  between  $t = 0$  and  $t_X$ , the variational lemma 4.1 implies that  $\text{sign } \mathbb{K}_2^\omega(u_\omega)$  also remains unchanged. Note that the case (b) of Lemma 4.1 is precluded by  $d_\omega(\varphi) < \delta_V$ , since it implies  $\mathbb{E}^\omega(\varphi) > \mathbb{E}^\omega(Q_\omega)/2 \geq \delta_U/2$  if  $\delta_V$  is small enough. Hence  $\text{sign } \mathbb{K}_2^\omega(u_\omega) = \text{sign } b_1(0)$ , so (ii) and (iii) define the same value of  $\mathfrak{S}_\omega$  for  $\varphi$ . Therefore  $\mathfrak{S}_\omega$  is well defined and continuous.

To show the invariance with respect to  $\omega$ , let  $\varphi \in H_r^1(\mathbb{R}^3)$  satisfy  $S_{\omega^\sharp} \varphi \in \check{\mathcal{H}}_{\omega^\sharp}$ . Let  $u$  be the solution of (1.1) with  $u(0) = \varphi$  and let  $u^\sharp := S_{\omega^\sharp} u$ . Then  $u^\sharp(t/\omega^\sharp)$  is the solution of (3.2) with  $u^\sharp(0) = S_{\omega^\sharp} \varphi$  and  $\omega = \omega^\sharp$ .

Suppose that  $\mathfrak{S}_{\omega^\sharp}(u^\sharp(0)) \neq \mathfrak{S}_{\omega^\sharp}(u^\sharp(0))$  and let  $I \ni 0$  be the maximal time interval where  $u^\sharp$  remains in  $\check{\mathcal{H}}_{\omega^\sharp}$  for both  $j = 0$  and  $j = 1$ . The discrepancy of  $\mathfrak{S}_\omega$  implies that either  $u^\sharp(t)$  or  $u^\sharp(t)$  is in the case (ii) at each  $t \in I$ , since  $\mathbb{M}\mathbb{H}^0(S_\omega \varphi)$  and  $\mathbb{K}_2^\omega(S_\omega \varphi) = \mathbb{K}_2(\varphi)$  are independent of  $\omega$ . Suppose that  $u^\sharp(0)$  is in the case (ii). By the ejection lemma as above,  $u^\sharp$  exits (ii) into the region (iii) either forward or backward in time. Meanwhile,  $u^\sharp$  must either enter the region (ii) or exit  $\check{\mathcal{H}}_{\omega^\sharp}$ . Since the solution does not blow up, exiting  $\check{\mathcal{H}}_{\omega^\sharp}$  is possible only through the region (ii).



Since  $\{t \in I \mid \text{(ii)}\}$  is open for each solution, we deduce that at some  $t = t_0 \in I$  both  $u^0$  and  $u^1$  are in the case (ii).

Decompose  $u^i(t_0)$  around  $Q_{\omega^i}$  as before

$$v^i := e^{-i\theta^i} u^i(t_0) - Q_{\omega^i} \in \mathcal{V}_{\omega^i}, \quad b_1^i := P_1^{\omega^i} v^i, \quad (4.14)$$

then  $d_{\omega^i}(u^i(t_0)) \sim \|v^i\|_{H^1} \sim |b_1^i|$ . Using (4.9), we have

$$\begin{aligned} |\angle \log \omega^p| &\sim \|\angle \Psi[\omega^p]\|_{\dot{H}^{1/2}} \lesssim \sum_{j=0,1} \text{dist}_{\dot{H}^{1/2}}(u(t_0), \{e^{i\theta} \Psi[\omega^j]\}_\theta) \\ &= \sum_{j=0,1} \text{dist}_{\dot{H}^{1/2}}(u^j(t_0), Q_{\omega^j}) \lesssim |b_1^0| + |b_1^1|. \end{aligned} \quad (4.15)$$

Since  $v^i \in \mathcal{V}_{\omega^i}$ , we have

$$\begin{aligned} 0 &= \langle i v^0 | Q'_{\omega^0} \rangle = \langle i e^{i\angle \theta^p} S_{\omega^0/\omega^1}(Q_{\omega^1} + v^1) | Q'_{\omega^0} \rangle \\ &= -\sin \angle \theta^p \langle Q_{\omega^1} | Q'_{\omega^0} \rangle + O(|\angle \log \omega^p| + \|v^1\|_2), \end{aligned} \quad (4.16)$$

hence  $|\angle \theta^p| \lesssim |\angle \log \omega^p| + \|v^1\|_2 \lesssim |b_1^0| + |b_1^1|$ . Using that  $iQ_\omega, Q'_\omega \in \text{Ker } P_1^\omega$ , we have

$$\begin{aligned} b_1^0 &= P_1^{\omega^0} [e^{i\angle \theta^p} S_{\omega^0/\omega^1}(Q_{\omega^1} + v^1) - Q_{\omega^0}] \\ &= b_1^1 + O((|\angle \theta^p| + |\angle \log \omega^p|)|b_1^1| + |\angle \theta^p|^2 + |\angle \log \omega^p|^2). \end{aligned} \quad (4.17)$$

Then using  $\text{sign } b_1^0 \neq \text{sign } b_1^1$ , we obtain

$$|b_1^0| + |b_1^1| = |\angle b_1^p| \lesssim |\angle \theta^p|^2 + |\angle \log \omega^p|^2 + |b_1^1|^2 \lesssim (|b_1^0| + |b_1^1|)^2 \lesssim \delta_V^2 \ll 1, \quad (4.18)$$

which is a contradiction, if  $\delta_V$  is small enough. Therefore  $\mathfrak{S}_\omega$  is invariant for  $\omega$ .  $\square$

## 5. ONE-PASS LEMMA

Now we are ready to prove the key dynamical property that any solution can *not* pass closely by the first excited states *more than once*. In the proof below in the region  $\mathfrak{S}_\omega = +1$ , we will use the results and the arguments in [5], which requires smallness of  $\mathbb{M}(u)$ , or equivalently largeness of  $\omega$ . To be precise about it, we have

**Lemma 5.1.** *There is a constant  $\omega_* \in [\omega_*, \infty)$  such that for any  $\omega \geq \omega_*$ , every  $\varphi \in \mathfrak{S}_\omega^{-1}(\{+1\})$  satisfies all the small-mass conditions in [5]. Specifically, using the constants  $\mu_*$  in Lemma 2.6 and  $\mu_p, \mu_*$  in [5, Theorems 1.1 and 7.1], we have*

$$\mathbb{M}(S_\omega^{-1} \varphi) < \min(\mu_*, \mu_p, \mu_*). \quad (5.1)$$

*Proof.* Immediate from Lemma 4.3 and  $\mathbb{M}(S_\omega^{-1} \varphi) = \omega^{-1/2} \mathbb{M}(\varphi)$ .  $\square$

**Lemma 5.2.** *There is a constant  $\delta_* \in (0, \min(\delta_X, c_X^{-1/2} \varepsilon_S)]$  such that if  $u_\omega$  is a solution of (3.2) for some  $\omega \geq \omega_*$  on a maximal existence interval  $(T_-, T_+)$ , satisfying*

$$d_\omega(u_\omega(t_1)) < \delta, \quad \mathbb{A}^\omega(u_\omega) < \mathbb{A}^\omega(Q_\omega) + c_X \delta^2, \quad (5.2)$$

*for some  $\delta \in (0, \delta_*]$  at some  $t_1 \in (T_-, T_+)$ , then there exists  $t_2 \in (t_1, T_+]$  such that  $d_\omega(u_\omega(t)) < \delta$  for  $t_1 \leq t < t_2$  and  $d_\omega(u_\omega(t)) > \delta$  for  $t_2 < t < T_+$ . If  $t_2 = T_+$ , then the trapping lemma 3.5 applies to  $u_\omega$ .*

The rest of this section is devoted to proving the above lemma. The solution  $u_\omega$  of (3.2) is fixed, so that we can abbreviate  $d(t) := d_\omega(u_\omega)$ , but all estimates will be uniform with no dependence on the particular choice of  $u_\omega$ .

The last sentence of the lemma is obvious from  $\delta \leq \delta_* \leq \delta_X$ . For a proof of the rest and main part of the lemma, it suffices to derive a contradiction from the following: Suppose that for some  $t_- < t_+$  within  $(T_-, T_+)$ ,

$$\max_{t \in [t_-, t_+]} d(t) > \min_{t \in [t_-, t_+]} d(t) = d(t_\pm) =: \delta \in (0, \delta_*]. \quad (5.3)$$

Taking  $\delta_* \leq c_X^{-1/2} \varepsilon_S$  ensures that  $u_\omega(t)$  stays in  $\check{\mathcal{H}}_\omega$  for  $t \in [t_-, t_+]$ , because

$$\mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega) < c_X \delta^2 \leq c_X \delta_*^2 \leq \varepsilon_S^2. \quad (5.4)$$

Hence  $\sigma := \mathfrak{S}_\omega(u_\omega(t)) \in \{\pm 1\}$  is independent of  $t \in [t_-, t_+]$ .

Taking  $\delta_* \ll \delta_V$ , decompose the time interval  $[t_-, t_+]$  as follows. Let  $\mathcal{M}$  be the set of all minimal points of  $d : [t_-, t_+] \rightarrow [\delta, \infty)$  with the minima less than  $\delta_V$ . Then applying the ejection lemma 3.3 from each  $t_0 \in \mathcal{M}$  forward and backward in time, we obtain a closed interval  $I(t_0) \subset [t_-, t_+]$  such that  $d(t)^2$  is strictly convex on  $I(t_0)$  with the unique minimal point  $t = t_0$  with  $d(t) = \delta_X$  on  $\partial I(t_0) \setminus \{t_\pm\}$ , and

$$e^{\alpha_\omega |t-t_0|} d(t_0) \sim d(t) \sim \sigma b_1(t) \sim \sigma \mathbb{K}_2^\omega(u_\omega(t)) + C_K \|\zeta(t_0)\|_{H^1} \quad (5.5)$$

on  $I(t_0)$ . The convexity on each  $I(t_0)$  implies that those intervals are mutually disjoint. Putting

$$I_H := \bigcup_{t_0 \in \mathcal{M}} I(t_0), \quad I_V := [t_-, t_+] \setminus I_H, \quad (5.6)$$

we have  $d \in [\delta, \delta_X]$  on  $I_H$  and  $d \in [\delta_V, \infty)$  on  $I_V$ . Lemma 2.3 (3) implies that  $(\mathbb{M} + \omega \mathbb{H}^0)(u_\omega) > C_M$  on  $[t_-, t_+]$ , since otherwise  $\mathbb{H}^0(u_\omega(t_\pm)) < \mathbb{H}^0(Q)/4 < \mathbb{H}^0(Q_\omega)/2$  contradicts that  $d(t_\pm) \leq \delta_* \ll 1$ . Then the variational lemma 4.1 implies

$$t \in I_V \implies \sigma \mathbb{K}_2^\omega(u_\omega) \geq \kappa_V(\delta_V) > 0. \quad (5.7)$$

Note that the case (b) of Lemma 4.1 is also precluded by the proximity to  $Q_\omega$ , which implies  $\mathbb{E}^\omega(u_\omega) > \mathbb{E}^\omega(Q_\omega)/2 \geq \delta_U/2$ .

**5.1. Blow-up region.** For  $\sigma = -1$ , we use a localized virial as in [7, §4.1], [8]

$$\mathcal{V}_m(t) := \langle m \phi_m u_\omega | i \partial_r u_\omega \rangle, \quad (5.8)$$

where  $\phi(r)$  is a smooth non-decreasing function satisfying

$$\phi(r) = \begin{cases} r & (r \leq 1) \\ 3/2 & (r \geq 2), \end{cases} \quad (5.9)$$

and  $\phi_m(r) = \phi(r/m)$  for some cut-off radius  $m > 1$  to be chosen shortly. Using the equation (3.2), we have

$$\begin{aligned} \dot{\mathcal{V}}_m &= 2\mathbb{K}_2^\omega(u_\omega) - \langle |\partial_r u_\omega|^2 | 2f_{0,m} \rangle + \langle |u_\omega/r|^2 | f_{1,m} \rangle \\ &\quad + \langle |u_\omega|^4 | f_{2,m} \rangle + 2[f_{0,m} \mathcal{S}'_\infty V^\omega](u_\omega), \end{aligned} \quad (5.10)$$

where  $f_{j,m}(r) := f_j(r/m)$  with

$$f_0 := 1 - \phi_r, \quad f_1 := -r^2 \Delta(\partial_r/2 + 1/r)\phi, \quad f_2 := 3/2 - (\partial_r/2 + 1/r)\phi. \quad (5.11)$$

The last term in (5.10) is the only essential difference from the case [7] without the potential. Since  $\mathcal{S}'_\infty V \in L^2 + L^\infty_0$ , for any  $\eta > 0$  there exists  $B(\eta) \in [1, \infty)$  and a decomposition  $\mathcal{S}'_\infty V = W_2 + W_\infty$  such that

$$\|W_\infty\|_\infty \leq \eta, \quad \|W_2\|_2 \leq B(\eta), \quad (5.12)$$

cf. [7, Lemma 2.1]. Let  $W_p^\omega := \omega^{-1}W_p(\omega^{-1/2}x)$ . Then

$$\begin{aligned} |[\mathcal{S}'_\infty V^\omega](u_\omega)| &\leq \|W_\infty^\omega\|_\infty \|u_\omega\|_{L^2(f_{0,m}dx)}^2 + \|W_2^\omega\|_2 \|f_{0,m}|u_\omega|^2\|_2 \\ &\leq \omega^{-1}\eta \|u_\omega\|_{L^2(|x|>m)}^2 + \omega^{-1/4}B(\eta) \|u_\omega\|_{L^4(f_{0,m}dx)}^2, \end{aligned} \quad (5.13)$$

where we used that  $\text{supp } f_{0,m} \subset \{|x| > m\}$  and  $0 \leq f_{0,m} \leq 1$ . The last  $L^4$  norm is treated in the same way as [7, (4.14)] by the radial Sobolev for  $\varphi \in H_r^1(\mathbb{R}^3)$

$$\begin{aligned} \|\varphi\|_{L^4(f_{0,m}dx)}^4 &\sim \int_m^\infty f'_{0,m}(s) \|\varphi\|_{L^4(|x|>s)}^4 ds \\ &\lesssim \int_m^\infty f'_{0,m}(s) s^{-2} \|\varphi\|_{L^2(|x|>s)}^3 \|\varphi_r\|_{L^2(|x|>s)} ds \\ &\leq m^{-2} \|\varphi\|_{L^2(|x|>m)}^3 \left[ \int_m^\infty f'_{0,m}(s) \|\varphi_r\|_{L^2(|x|>s)}^2 ds \cdot \int_m^\infty f'_{0,m}(s) ds \right]^{1/2} \\ &\sim m^{-2} \|\varphi\|_{L^2(|x|>m)}^3 \|\varphi_r\|_{L^2(f_{0,m}dx)}. \end{aligned} \quad (5.14)$$

The same estimate applies to the second last term of (5.10), because  $|f_{2,m}| \lesssim f_{0,m}$ . The norm  $\|\partial_r u_\omega\|_{L^2(f_{0,m}dx)}$  can be absorbed by the second term on the right of (5.10) after Young. Using also that  $\|u_\omega\|_{L^2(|x|>m)} \lesssim \|Q_\omega\|_{L^2} + \delta \lesssim 1$ , we obtain

$$\dot{\mathcal{V}}_m + 2\mathbb{K}_2^\omega(u_\omega) \lesssim \omega^{-1}\eta + m^{-2} + (\omega^{-1/4}m^{-1}B(\eta))^{4/3}. \quad (5.15)$$

Hence, choosing  $\eta$  small and then  $m$  large such that

$$\eta \ll \omega_* \kappa_V(\delta_V), \quad m \gg \max(\kappa_V(\delta_V)^{-1/2}, \omega_*^{-1/4}B(\eta)\kappa_V(\delta_V)^{-3/4}), \quad (5.16)$$

we have  $\dot{\mathcal{V}}_m \leq -\kappa_V(\delta_V) < 0$  on  $I_V$ . On each  $I(t_0)$  in  $I_H$ , we have

$$\|u_\omega\|_{L^2(|x|>m)} \leq \|Q_\omega\|_{L^2(|x|>m)} + \|v\|_{L^2(|x|>m)} \lesssim m^{-1} + d(t), \quad (5.17)$$

and so, using the hyperbolic estimate on  $\mathbb{K}_2^\omega$  in Lemma 3.3 as well,

$$\begin{aligned} [\sigma \dot{\mathcal{V}}_m]_{\partial I(t_0)} &= \int_{I(t_0)} \sigma \dot{\mathcal{V}}_m dt \gtrsim \int_{I(t_0)} (e^{\alpha_\omega|t-t_0|} - 2C_K) d(t_0) - O(m^{-2}) dt \\ &\gtrsim \delta_X - Cm^{-2} \log(\delta_X/\delta). \end{aligned} \quad (5.18)$$

On the other hand,  $d(t) = \delta$  at  $t = t_\pm$  and  $\|xQ_\omega\|_2 + \|x\nabla Q_\omega\|_2 \lesssim 1$  imply

$$|[\mathcal{V}_m]_{t_-}^{t_+}| \lesssim \delta + m\delta^2. \quad (5.19)$$

We can choose  $m = 1/\delta$  satisfying (5.16) if  $\delta_*$  is so small that

$$\delta_* \leq C^{-1} \kappa_V(\delta_V)^{1/2} \min(1, (\omega_* \kappa_V(\delta_V))^{1/4} B(C\omega_* \kappa_V(\delta_V))^{-1}) \quad (5.20)$$

for some large constant  $C \in (1, \infty)$ . Then we have

$$|[\mathcal{V}_m]_{t_-}^{t_+}| \lesssim \delta \ll \delta_X \lesssim \int_{t_-}^{t_+} \sigma \dot{\mathcal{V}}_m dt, \quad (5.21)$$

which is a contradiction. Therefore (5.3) is impossible in the case  $\sigma = -1$ .

**5.2. Scattering region.** For  $\sigma = +1$ , we could argue as in [7], which would however suffer from the loss of sign in the localized virial due to the potential or the ground states, which happens as the solution is expected to be very dispersed in the variational time  $I_V$ . Specifically, the argument would fail at [7, (4.31)]. Then one option to overcome it would be to estimate possible dispersion and propagation along any returning orbit so that we can find an appropriate cut-off radius  $m$ . Instead of that, we rely on the minimal contradiction argument of Kenig-Merle [4] using the profile decomposition in [5], to show that there is a positive lower bound on  $\delta$  for which the return path (5.3) can exist.

Let  $\omega_n \geq \omega_*$  be a sequence such that  $\omega_n \rightarrow \varpi \in [\omega_*, \infty]$  and let  $\tilde{u}_n$  be a sequence of solutions to the rescaled equation (3.2) with  $\omega = \omega_n$  and (5.3) at some  $t_{\pm,n}$  with

$$\delta_X \gg \delta = \delta_n \rightarrow 0, \quad \mathfrak{S}_{\omega_n}(\tilde{u}_n(t_{\pm,n})) = +1. \quad (5.22)$$

After appropriate translation of each  $\tilde{u}_n$  in time, there exist sequences  $\tilde{R}_n < 0 < \tilde{S}_n < \tilde{T}_n$  such that, abbreviating  $d_n(t) := d_{\omega_n}(\tilde{u}_n(t))$  and  $\alpha_n := \alpha_{\omega_n}$ ,

$$\begin{aligned} d_n(\tilde{R}_n) &= \delta_n = d_n(\tilde{T}_n), \quad d_n(0) = \delta_X = d_n(\tilde{S}_n), \\ \tilde{R}_n \leq t \leq 0 &\implies d_n(t) \sim e^{\alpha_n t} \delta_X, \\ \tilde{R}_n < t < \tilde{T}_n &\implies d_n(t) > \delta_n, \\ \tilde{S}_n \leq t \leq \tilde{T}_n &\implies d_n(t) \sim e^{-\alpha_n(t-\tilde{S}_n)} \delta_X. \end{aligned} \quad (5.23)$$

Since  $\tilde{u}_n$  stays in  $\check{\mathcal{H}}_{\omega_n}$  with  $\mathfrak{S}_{\omega_n} = +1$ , by Lemma 4.3 it is uniformly bounded in  $H^1$  on  $[\tilde{R}_n, \tilde{T}_n]$ . Let  $u_n := S_{\omega_n}^{-1} \tilde{u}_n(\omega_n t)$  be the sequence of unscaled solutions, and

$$(R_n, S_n, T_n) := \omega_n^{-1}(\tilde{R}_n, \tilde{S}_n, \tilde{T}_n). \quad (5.24)$$

Since  $\omega_n \geq \omega_*$ , Lemma 5.1 allows us to apply the arguments in [5] to  $u_n$ . Using the coordinate around the ground solitons as in [5, (4.9)], we can decompose

$$u_n(t) = \Phi[z_n(t)] + \eta_n(t) = \Phi[z_n(t)] + R[z_n(t)]\xi_n(t) \quad (5.25)$$

such that  $\eta_n(t) \in \mathcal{H}_c[z_n(t)]$  and  $\xi_n(t) \in P_c(H_r^1)$  for  $t \in [R_n, T_n]$ , where

$$\mathcal{H}_c[z] := \{\varphi \in H_r^1 \mid 0 = \langle i\varphi | \partial_z \Phi[z] \rangle = \langle i\varphi | \partial_{\bar{z}} \Phi[z] \rangle\}, \quad R[z] = (P_c|_{\mathcal{H}_c[z]})^{-1}. \quad (5.26)$$

Let  $C_6 > 0$  be the best constant such that  $\inf_{\theta} \|e^{i\theta} Q_{\omega} - \varphi\|_6 \leq C_6 d_{\omega}(\varphi)$  holds for all  $\omega \geq \omega_*$  and  $\varphi \in H_r^1$ , and let

$$\delta_W := \inf\{\|e^{i\theta} Q_{\omega} - S_{\omega} \Phi[z]\|_6 / C_6 \mid \omega \geq \omega_*, \theta \in \mathbb{R}, z \in Z_*\}. \quad (5.27)$$

Then  $\delta_W > 0$  because both  $\{e^{i\theta} Q_{\omega}\}_{\omega \geq \omega_*, \theta \in \mathbb{R}}$  and  $\{S_{\omega} \Phi[z]\}_{z \in Z_*}$  are precompact in  $H^1$  and the normand is never 0 even on their closures. Hence for large  $n$ , there exists  $\tilde{S}'_n \in [\tilde{S}_n, \tilde{T}_n]$  such that  $d_n(\tilde{S}'_n) = \min(\delta_X, \delta_W/2)$ . Then using the scale invariance of  $\mathfrak{st} = L_t^4 L^6$ , we obtain

$$\begin{aligned} \|\xi_n\|_{\mathfrak{st}(S_n, T_n)} &\sim \|\eta_n\|_{\mathfrak{st}(S_n, T_n)} \geq \|\tilde{u}_n - S_{\omega_n} \Phi[z_n(t/\omega_n)]\|_{\mathfrak{st}(\tilde{S}'_n, \tilde{T}_n)} \\ &\geq C_6 |\tilde{T}_n - \tilde{S}'_n|^{1/4} (\delta_W - d_n(\tilde{S}'_n)) \\ &\sim C_6 \delta_W \log^{1/4}(\delta_W/\delta_n) \rightarrow \infty \quad (n \rightarrow \infty), \end{aligned} \quad (5.28)$$

because of the exponential behavior on  $[S_n, T_n]$  in (5.23). Similarly we have  $\tilde{R}_n \lesssim -\log(\delta_X/\delta_n) \rightarrow -\infty$  and  $\tilde{T}_n \gtrsim \log(\delta_X/\delta_n) \rightarrow \infty$ . Since  $\tilde{u}_n$  are uniformly bounded in  $C([\tilde{R}_n, \tilde{T}_n]; H_r^1)$ , a standard weak compactness argument implies that, passing to a subsequence,  $\tilde{u}_n$  converges to some  $\tilde{u}_\infty$  in  $C(\mathbb{R}; w\text{-}H_r^1) \cap L^\infty(\mathbb{R}; H^1)$ , which solves the limit equation, that is (3.2) with  $\omega = \varpi < \infty$  or (1.9) if  $\varpi = \infty$ .

The weak convergence implies  $\mathbb{E}^\varpi(\tilde{u}_\infty) \leq \mathbb{E}^\varpi(Q_\varpi)$  and  $\mathbb{M}(\tilde{u}_\infty) \leq \mathbb{M}(Q)$ , as well as  $d_\varpi(\tilde{u}_\infty(t)) \lesssim e^{\alpha_\varpi t} \delta_X$  for all  $t < 0$ , hence by the conservation law

$$(\mathbb{M}, \mathbb{E}^\varpi)(\tilde{u}_\infty) = (\mathbb{M}, \mathbb{E}^\varpi)(Q_\varpi). \quad (5.29)$$

Therefore the convergence  $\tilde{u}_n \rightarrow \tilde{u}_\infty$  is strong in  $H^1$ , locally uniformly in  $t$ .

**5.2.1. The case of bounded time frequency  $\varpi < \infty$ .** In this case, the above strong convergence is translated to that of  $u_n$  to  $u_\infty(t) := S_\varpi^{-1} \tilde{u}_\infty(\varpi t)$ . Apply the profile decomposition in [5, §5–7] to  $\xi_n$  on  $[0, T_n]$ . Then the strong convergence of  $u_n(0)$  implies that there is only one nonlinear profile, which is the strong limit at  $t = 0$ , and the remainder is strongly vanishing in  $H^1$ . Hence [5, Theorem 7.2] and (5.28) imply that  $u_\infty$  does not scatter to  $\Phi$  as  $t \rightarrow \infty$ .

Then the main result of [5] below  $\mathcal{S}_1$  together with (5.29) implies that  $u_\infty$  is a minimal non-scattering solution, so the argument in [5, §8] implies that  $u_\infty(0, \infty) \subset H^1$  is precompact. Since

$$d_\infty(t) := d_\varpi(\tilde{u}_\infty(t)) \quad (5.30)$$

is exponentially decaying as  $t \rightarrow -\infty$ , the trapping lemma 3.5 implies that  $\tilde{u}_\infty(t) = e^{i(a-t)} Q_\varpi + o(1)$  in  $H^1$  as  $t \rightarrow -\infty$  for some  $a \in \mathbb{R}$ . In particular, the entire trajectory  $\tilde{u}_\infty(\mathbb{R})$  is precompact in  $H^1$ .

Now we use another localized virial identity as in [7, §4.2]

$$\begin{aligned} \mathcal{V}_m &:= \langle \psi_m \tilde{u}_\infty | i\mathcal{S}'_2 \tilde{u}_\infty \rangle, \\ \dot{\mathcal{V}}_m &= 2\mathbb{K}_2^\varpi(\psi_m \tilde{u}_\infty) + \langle |\tilde{u}_\infty/m|^2 | f_{3,m} \rangle - \langle |\tilde{u}_\infty|^4 | f_{4,m} \rangle \\ &\quad - 2m^{-1} [r\mathcal{S}'_\infty V^\varpi](\psi_m \tilde{u}_\infty), \end{aligned} \quad (5.31)$$

where  $\psi_m(r) = \psi(r/m)$  and  $f_{j,m}(r) = f_j(r/m)$  as before, with

$$\psi := (1+r)^{-1}, \quad f_3 := (1+r)^{-4}, \quad f_4 := (1+r)^{-4} r(r^2 + 7r/2 + 4). \quad (5.32)$$

The last term in (5.31) is bounded by

$$|m^{-1} [r\mathcal{S}'_\infty V^\varpi](\psi_m \tilde{u}_\infty)| \lesssim [\min(r/m, m/r) |\mathcal{S}'_\infty V^\varpi|](\tilde{u}_\infty). \quad (5.33)$$

Combining the precompactness with the above estimate, as well as the decay of  $f_{3,m}$  and  $f_{4,m}$ , yields some  $m \geq 1$  such that for all  $t \in \mathbb{R}$

$$|\dot{\mathcal{V}}_m - 2\mathbb{K}_2^\varpi(\tilde{u}_\infty)| \ll \kappa_V(\delta_V). \quad (5.34)$$

Let  $0 < \delta \ll \min(1/m, \delta_V)$ ,  $d_\infty(t_-) = \delta$  with  $t_- < 0$  and

$$t_+ := \inf\{t > t_- \mid d_\infty(t) \leq \delta\}, \quad (5.35)$$

then  $t_+ > t_-$  because  $\partial_t d_\infty(t_-) > 0$  by the ejection lemma 3.3. Suppose that  $t_+ < \infty$  for contradiction. Decompose  $[t_-, t_+] = I_H \cup I_V$  as in (5.6). Using the exponential decay of  $Q_\varpi$ , we have on each  $I(t_0)$

$$\dot{\mathcal{V}}_m = 2\mathbb{K}_2^\varpi(\tilde{u}_\infty) + O(m^{-1} + d_\infty^2) \gtrsim (e^{\alpha\varpi|t-t_0|} - 2C_K)d_\infty(t_0) - O(m^{-1}), \quad (5.36)$$

and so

$$[\mathcal{V}_m]_{\partial I(t_0)} \gtrsim \delta_X - Cm^{-1} \log(\delta_X/\delta) \geq \delta_X(1 - \delta/\delta_X \log(\delta_X/\delta)) > \delta_X/2. \quad (5.37)$$

On the other hand, (5.7) and (5.34) imply that  $\dot{\mathcal{V}}_m \geq \kappa_V(\delta_V) > 0$  on  $I_V$ . Hence

$$\delta_X \lesssim [\mathcal{V}_m]_{t_-}^{t_+} \lesssim \delta + m\delta^2 \lesssim \delta \ll \delta_X, \quad (5.38)$$

leading to a contradiction. Therefore  $t_+ = \infty$ , which implies however that  $\mathcal{V}_m \rightarrow \infty$  as  $t \rightarrow \infty$  by the above argument on  $[t_-, \infty)$ , contradicting the precompactness of  $\tilde{u}_\infty(\mathbb{R})$ . Thus we have precluded the case  $\varpi < \infty$ .

**5.2.2. The concentrating case  $\varpi = \infty$ .** In this case, we have  $Q_\varpi = Q$ , and the limit  $\tilde{u}_\infty$  is a global solution of (1.9) exponentially convergent to  $\{e^{i\theta}Q\}_\theta$  as  $t \rightarrow -\infty$ . Then the classification by [1] implies that  $\tilde{u}_\infty$  is, modulo time translation, either the soliton  $e^{-it}Q$  itself or the unique solution  $w_+$  which is exponentially converging to  $e^{-it}Q$  as  $t \rightarrow -\infty$  and scattering to 0 as  $t \rightarrow +\infty$ . The strong convergence at  $t = 0$  implies  $d_\infty(\tilde{u}_\infty(0)) = \delta_X > 0$ , precluding the soliton case. Hence  $\tilde{u}_\infty = w_+$ . If  $\tilde{S}_n$  converges to some finite  $\tilde{S}_\infty < \infty$  along a subsequence, then  $d_\infty(\tilde{u}_\infty(t)) \lesssim e^{-\alpha(t-\tilde{S}_\infty)}\delta_X$  for  $t \geq \tilde{S}_\infty$ , contradicting the scattering to 0 of  $\tilde{u}_\infty$  as  $t \rightarrow \infty$ . Hence  $\tilde{S}_n \rightarrow \infty$ .

The scattering to 0 implies that for any  $\nu > 0$  there exists  $\tilde{\tau} > 0$  such that

$$\|e^{-i(t-\tilde{\tau})\Delta}\tilde{u}_\infty(\tilde{\tau})\|_{\text{st}(\tilde{\tau},\infty)} \leq \nu. \quad (5.39)$$

Then, putting  $\tau_n := \omega_n^{-1}\tilde{\tau}$  and using  $\tilde{u}_n(\tilde{\tau}) \rightarrow \tilde{u}_\infty(\tilde{\tau})$  in  $H^1$ , we obtain

$$\begin{aligned} \|e^{-i(t-\tau_n)\Delta}u_n(\tau_n)\|_{\text{st}(\tau_n,\infty)} &= \|e^{-i(t-\tilde{\tau})\Delta}\tilde{u}_n(\tilde{\tau})\|_{\text{st}(\tilde{\tau},\infty)} \\ &= \|e^{-i(t-\tilde{\tau})\Delta}\tilde{u}_\infty(\tilde{\tau})\|_{\text{st}(\tilde{\tau},\infty)} + o(1) < 2\nu \end{aligned} \quad (5.40)$$

for large  $n$ . We also have uniform bounds

$$\|u_n(\tau_n)\|_{H^\theta} \lesssim \omega_n^{(\theta-1/2)/2} \|\tilde{u}_n(\tilde{\tau})\|_{H^\theta} \lesssim \omega_n^{(\theta-1/2)/2} \quad (5.41)$$

for  $0 \leq \theta \leq 1$ , hence in particular,

$$|z_n(\tau_n)| \lesssim \|u_n(\tau_n)\|_2 \lesssim \omega_n^{-1/4}. \quad (5.42)$$

Let  $\xi_n^0 := e^{-i(t-\tau_n)\Delta}\xi_n(\tau_n)$ . Using

$$\xi_n(\tau_n) = P_c(u_n(\tau_n) - \Phi[z_n(\tau_n)]), \quad (5.43)$$

and

$$|(\phi_0|u_n(\tau_n) - \Phi[z_n(\tau_n)])| \lesssim \|u_n(\tau_n)\|_2 + |z_n(\tau_n)| \lesssim \omega_n^{-1/4}, \quad (5.44)$$

we have by the free Strichartz estimate,

$$\begin{aligned} \|\xi_n^0\|_{\text{st}(\tau_n,\infty)} &\leq \|e^{-i(t-\tau_n)\Delta}u_n(\tau_n)\|_{\text{st}(\tau_n,\infty)} + C\omega_n^{-1/4} < 3\nu, \\ \|\xi_n^0\|_{L_t^4 L^3(\tau_n,\infty)} &\lesssim \|\xi_n(\tau_n)\|_2 \lesssim \omega_n^{-1/4} \ll \nu, \end{aligned} \quad (5.45)$$

for large  $n$ . Let  $\xi_n^1$  be the linearized solution with the same initial data, namely

$$(i\partial_t + H - B[z_n])\xi_n^1 = 0, \quad \xi_n^1(\tau_n) = \xi_n(\tau_n) \in P_c(H_r^1), \quad (5.46)$$

where  $B[z]$  is the  $\mathbb{R}$ -linear operator defined by

$$B[z]\varphi = P_c\{2|\Phi[z]|^2 R[z]\varphi + \Phi[z]^2 \overline{R[z]\varphi}\}. \quad (5.47)$$

Then we have

$$(i\partial_t + H - B[z_n])P_c\vartriangleleft\xi_n^\triangleright = -P_c V\xi_n^0 + B[z_n]P_c\xi_n^0, \quad \vartriangleleft\xi_n^\triangleright(\tau_n) = 0. \quad (5.48)$$

Applying the non-admissible Strichartz [7, (4.41)] to (5.48), we obtain

$$\begin{aligned} \|P_c\vartriangleleft\xi_n^\triangleright\|_{\text{st}(\tau_n, \infty)} &\lesssim \|V\xi_n^0 - B[z_n]P_c\xi_n^0\|_{L_t^4 L^{6/5}(\tau_n, \infty)} \\ &\lesssim [\|V\|_{L^2} + |z_n|] \|\xi_n^0\|_{L_t^4 L^3(\tau_n, \infty)} \ll \nu. \end{aligned} \quad (5.49)$$

Adding it to (5.45), and using  $\xi_n^1 = P_c\xi_n^1$ , we obtain

$$\|\xi_n^1\|_{\text{st}(\tau_n, \infty)} < 4\nu \quad (5.50)$$

for large  $n$ . Hence, taking  $\nu > 0$  small, and using (5.41) as well, we deduce from the small data scattering [5, Lemma 6.2] that  $u_n$  scatters to  $\Phi$  as  $t \rightarrow \infty$  with a uniform bound for large  $n$

$$\|\xi_n\|_{\text{st}(\tau_n, \infty)} \lesssim \nu. \quad (5.51)$$

Since  $\tilde{S}_n \rightarrow \infty$  implies  $\tau_n < S_n$  for large  $n$ , the above bound contradicts (5.28). Therefore  $\varpi = \infty$  is also impossible, which means that there can not exist such a sequence of solutions  $\tilde{u}_n$  in (5.23). Thus we finish the proof of Lemma 5.2.  $\square$

*Remark 5.3.* The additional assumption  $V \in L^2(\mathbb{R}^3)$  is needed only in the above estimate (5.49) and similarly in (6.39). It could be replaced with the following statement: For any bounded sequence  $\varphi_n$  in  $\dot{H}^{1/2}(\mathbb{R}^3)$ , we have

$$\|e^{-it\Delta}\varphi_n\|_{\text{st}(0, \infty)} + \|\varphi_n\|_2 \rightarrow 0 \implies \|e^{-itH}P_c\varphi_n\|_{\text{st}(0, \infty)} \rightarrow 0, \quad (5.52)$$

$V \in L^2(\mathbb{R}^3)$  is a sufficient condition, as shown above by the Strichartz perturbation, but the latter does not work if we merely assume  $V \in L^2 + L_0^\infty$ .

## 6. DYNAMICS AWAY FROM THE EXCITED STATES

The one-pass lemma 5.2 ensures that if a solution of the rescaled NLS (3.2) leaves the small neighborhood of  $\mathcal{Q}_\omega$ , then it never returns. In this section, we investigate behavior of such solutions  $u_\omega$  staying away from  $\mathcal{Q}_\omega$ , after some time or for all time. More precisely, let  $u_\omega$  be any solution of (3.2) for some  $\omega \geq \omega_*$  satisfying

$$\mathbb{A}^\omega(u_\omega) \leq \mathbb{A}^\omega(Q_\omega) + c_X \delta_*^2, \quad \inf_{0 \leq t < T_+} d_\omega(u_\omega(t)) \geq \delta_*, \quad (6.1)$$

where  $T_+ \in (0, \infty]$  is the maximal existence time of  $u_\omega$ . Thanks to (5.4), we have the same decomposition of  $[0, T_+) = I_H \cup I_V$  as in (5.6) with  $\delta = \delta_*$ , and the sign  $\sigma := \mathfrak{S}_\omega(u_\omega(t)) \in \{\pm 1\}$  remains constant for  $t \in [0, T_+)$ , which distinguishes the scattering and the blow-up cases.

**6.1. Blow-up region.** In the case  $\sigma = -1$ , the solution blows up.

**Lemma 6.1.** *For every  $\omega \geq \omega_*$  and every solution  $u_\omega$  of (3.2) satisfying*

$$\mathbb{A}^\omega(u_\omega) \leq \mathbb{A}^\omega(Q_\omega) + c_X \delta_*^2, \quad \inf_{0 \leq t < T_+} d_\omega(u_\omega(t)) \geq \delta_*, \quad \mathfrak{S}_\omega(u_\omega(0)) = -1, \quad (6.2)$$

where  $T_+$  is the maximal existence time, blows up in finite time, namely  $T_+ < \infty$ .

Using the localized virial estimate in Section 5.1, the proof is essentially the same as [7, §4.1] in the case without the potential, because the region  $\mathfrak{S}_\omega = -1$  is away from the ground states, where  $\mathbb{K}_2^\omega$  degenerates. The detail is omitted.

Also note that  $\omega \geq \omega_*$  is enough in this region, since the profile decomposition in [5] is not needed. In fact, the region of  $u_\omega(0)$  in the above lemma is unbounded in  $L^2(\mathbb{R}^3)$ , though it is not essentially new compared with [5], since those initial data with large  $L^2$  need very negative energy  $\mathbb{E}$  to satisfy (6.2), for which proving the blow-up is easier.

**6.2. Scattering region.** In the case  $\sigma = +1$ ,  $u_\omega$  scatters to  $\Phi$  as  $t \rightarrow \infty$ . The global existence is immediate from the  $H^1$  bound of Lemma 4.3, so the main part is to prove the scattering. As in [7], it is done by using the profile decomposition. In the same way as in the previous section, we need to distinguish between the case of bounded  $\omega$  and the case of  $\omega \rightarrow \infty$ .

For each  $\omega \geq \omega_*$  and  $A \leq c_X \delta_*^2$ , let  $\text{FS}_\omega(A)$  be the set of all the solutions  $u$  of (1.1) global in  $t > 0$  satisfying

$$\mathbb{A}^\omega(u_\omega) \leq \mathbb{A}^\omega(Q_\omega) + A, \quad \inf_{t \geq 0} d_\omega(u_\omega(t)) \geq \delta_*, \quad (6.3)$$

where  $u_\omega(t) := S_\omega u(t/\omega)$  is the rescaled solution of (3.2). In the original scale, the first condition is equivalent to, putting  $\mu := \mathbb{M}(\Psi[\omega])$ ,

$$\mathbb{A}_\omega(u) \leq \mathcal{E}_1(\mu) + \omega\mu + \omega^{1/2}A = \omega^{1/2}(\mathbb{A}(Q) + A + O(\omega^{-1/4})). \quad (6.4)$$

Now we look for a minimal solution in  $\text{FS}_\omega(A) \setminus \mathcal{S}$ . Note that (6.3) with  $A \leq c_X \delta_*^2$  implies that  $u_\omega = S_\omega u(t/\omega)$  stays in  $\tilde{\mathcal{H}}_\omega$  for all  $t \geq 0$ , and the case  $\mathfrak{S} = -1$  is precluded by the blow-up Lemma 6.1. Hence Lemma 5.1 with  $\omega \geq \omega_*$  allows us to apply the arguments in [5] to any solution  $u$  in  $\text{FS}_\omega(A)$ . Using the decomposition  $u(t) = \Phi[z(t)] + R[z(t)]\xi(t)$  of [5, Lemma 4.1], let

$$\begin{aligned} \text{ST}^\omega(A) &:= \sup_{u \in \text{FS}_\omega(A)} \|\xi\|_{\text{st}(0, \infty)}, \\ A_\omega^* &:= \sup\{A < c_X \delta_*^2 \mid \text{ST}^\omega(A) < \infty\}, \quad A^* := \inf_{\omega \geq \omega_*} A_\omega^*. \end{aligned} \quad (6.5)$$

Since the region  $\{(\mu, e) \in \mathbb{R}^2 \mid e + \omega\mu < \mathbb{A}_\omega(\Psi[\omega])\}$  is tangent from below to the graph  $e = \mathcal{E}_1(\mu)$  at  $\mu = \mathbb{M}(\Psi[\omega])$  because of  $\mathcal{E}_1'' > 0$ , the region  $\mathbb{A}^\omega(u_\omega) < \mathbb{A}^\omega(Q_\omega)$  for  $\omega \geq \omega_*$  is covered by the scattering below  $\mathcal{S}_1$  in [5]. Hence

$$A_\omega^* \geq 0 \quad (6.6)$$

for each  $\omega \geq \omega_*$ . Now suppose for contradiction that

$$A^* \ll c_X \delta_*^2. \quad (6.7)$$



Then there exist sequences  $\omega_n \geq \omega_*$ ,  $A_n > 0$ , and  $u_n \in \text{FS}_{\omega_n}(A_n)$  such that

$$\begin{aligned} \omega_n &\rightarrow \varpi \in [\omega_*, \infty], \quad c_X \delta_*^2 > A_n \rightarrow A^*, \\ u_n(t) &= \Phi[z_n(t)] + R[z_n(t)]\xi_n(t), \quad \|\xi_n\|_{\text{st}(0, \infty)} \rightarrow \infty. \end{aligned} \quad (6.8)$$

Let

$$\tilde{u}_n(t) := S_{\omega_n} u_n(t/\omega_n) \quad (6.9)$$

Since  $\mathfrak{S}_{\omega_n}(\tilde{u}_n) = +1$ , Lemma 4.3 implies that  $\tilde{u}_n(t)$  is uniformly bounded in  $H^1$ . By Lemmas 3.3 and 5.2, we may additionally impose, after translation in time,

$$d_{\omega_n}(\tilde{u}_n(0)) \geq \delta_X, \quad (6.10)$$

since if it cannot be achieved by translation, then the trapping lemma 3.5 applies to  $\tilde{u}_n$ , contradicting  $u_n \in \text{FS}_{\omega_n}(A_n)$  with  $A_n < c_X \delta_*^2$ .

6.2.1. *The case  $\varpi < \infty$ .* Apply the nonlinear profile decomposition of [5] to the sequence of solutions  $u_n$  on the time interval  $[0, \infty)$ . Here the procedure is outlined for the sake of notation. Let (after extracting a subsequence)

$$\xi_n^L = \sum_{0 \leq j < J} \lambda_n^j + \gamma_n^J \quad (6.11)$$

be the linearized profile decomposition of [5, Lemma 5.3], where  $\xi_n^L$ ,  $\lambda_n^j$  and  $\gamma_n^J$  solve the same linearized equation

$$(i\partial_t + H - B[z_n])\xi = 0, \quad (6.12)$$

with the initial conditions  $\xi_n^L(0) = \xi_n(0)$  and  $\lambda_n^j(s_n^j) = \text{w-}\lim_{m \rightarrow \infty} \xi_m^L(s_m^j)$  for some time sequences  $s_n^j \in [0, \infty)$  satisfying  $s_n^0 = 0$  and  $s_n^j - s_n^k \rightarrow \pm\infty$  as  $n \rightarrow \infty$  for  $0 \leq j < k < J$ . After fixing  $J$  large enough, let  $\Lambda_n^j$  be the nonlinear profiles, defined by the weak limit

$$\xi_\infty^j(t) = \text{w-}\lim_{n \rightarrow \infty} \xi_n(t + s_n^j), \quad \Lambda_n^j := \xi_\infty^j(t - s_n^j), \quad (6.13)$$

after passing to a further subsequence if necessary. Then by [5, Theorem 7.2], there exists at least one nonlinear profile which does not scatter as  $t \rightarrow \infty$ , since otherwise  $\|\xi_n\|_{\text{st}(0, \infty)}$  would be bounded. Let  $\xi_\infty^l$  be a nonlinear profile with  $\|\xi_\infty^l\|_{\text{st}(0, \infty)} = \infty$  which is minimal in the sense that if  $s_n^j - s_n^l \rightarrow -\infty$  then the nonlinear profile  $\xi_\infty^j$  scatters as  $t \rightarrow \infty$ . Let  $\mu_\infty := \mathbb{M}(\Psi[\varpi])$ ,

$$u_\infty^l(t) := \text{w-}\lim_{n \rightarrow \infty} u_n(t + s_n^l), \quad \tilde{u}_\infty^l(t) := S_\varpi u_\infty^l(t/\varpi). \quad (6.14)$$

Then  $\tilde{u}_\infty^l$  is a solution of (3.2) with  $\omega = \varpi$ , and the weak convergence  $u_n(s_n^l) \rightharpoonup u_\infty^l(0)$  in  $H_r^1$  implies

$$\mathbb{A}^\varpi(\tilde{u}_\infty^l) \leq \liminf_{n \rightarrow \infty} \mathbb{A}^{\omega_n}(\tilde{u}_n) = \mathbb{A}^\varpi(Q_\varpi) + A^*. \quad (6.15)$$

Suppose that  $d_\varpi(\tilde{u}_\infty^l(\tau)) \ll \delta_*$  at some  $\tau \in \mathbb{R}$  ( $\tau \geq 0$  if  $l = 0$ ), and put

$$t_n := \omega_n(s_n^l + \varpi^{-1}\tau), \quad \varphi_n := \tilde{u}_n(t_n) - \tilde{u}_\infty^l(\tau), \quad (6.16)$$

then  $\varphi_n \rightarrow 0$  weakly in  $H_r^1$ . Since  $d_{\omega_n}(\tilde{u}_n(t_n)) \gtrsim \delta_* \gg d_{\varpi}(\tilde{u}_{\infty}^l(\tau))$ , we deduce  $\|\varphi_n\|_{H^1} \gtrsim \delta_*$ , then (6.15) with the weak convergence implies

$$\mathbb{A}^{\varpi}(Q_{\varpi}) + A^* \geq \mathbb{A}^{\varpi}(\tilde{u}_{\infty}^l) + \lim_{n \rightarrow \infty} \|\varphi_n\|_{H^1}^2/2 \geq \mathbb{A}^{\varpi}(Q_{\varpi}) + c\delta_*^2, \quad (6.17)$$

for some constant  $c > 0$ , contradicting  $A^* \ll \delta_*^2$ .

Therefore  $\delta_* \lesssim d_{\varpi}(\tilde{u}_{\infty}^l(t))$  at all  $t$ , and by Lemma 3.5, some translate of  $\tilde{u}_{\infty}^l$  in  $t$  belongs to  $\text{FS}_{\varpi}(A^*)$ . Since  $\|\xi_{\infty}^l\|_{\text{st}(0,\infty)} = \infty$ , the definition of  $A^*$  implies that (6.15) must be equality, hence  $u_n(s_n^l) \rightarrow u_{\infty}^l(0)$  strongly in  $H^1$ , thus we have obtained a minimal element  $u(t) := u_{\infty}^l(t+T)$  for some  $T \geq 0$ , satisfying

$$u = \Phi[z] + R[z]\xi \in \text{FS}_{\varpi}(A^*), \quad \|\xi\|_{\text{st}(0,\infty)} = \infty. \quad (6.18)$$

Next we prove that for such a critical element  $u \in \text{FS}_{\varpi}(A^*)$  with  $\|\xi\|_{\text{st}(0,\infty)} = \infty$ , the orbit  $\{u(t) \mid t \geq 0\}$  is precompact in  $H_r^1$ . For any sequence  $t_n \rightarrow \infty$ , the same argument as above applies to the sequence of solutions  $u_n := u(t+t_n)$  on  $[-t_n, 0]$  and on  $[0, \infty)$ , because  $\|\xi\|_{\text{st}(0,t_n)} \rightarrow \infty$  and  $\|\xi\|_{\text{st}(t_n,\infty)} = \infty$  as  $n \rightarrow \infty$ . Consider the profile decomposition with the first nonlinear profile

$$\xi_{\infty}^0(t) = \text{w-}\lim_{n \rightarrow \infty} \xi_n(t) = \text{w-}\lim_{n \rightarrow \infty} \xi(t+t_n), \quad (6.19)$$

after extracting a subsequence. Let  $u_{\infty}^0$  be the weak limit of  $u_n(t) = u(t+t_n)$ .

If  $\xi_{\infty}^0$  is not scattering as  $t \rightarrow \infty$ , then it is a non-scattering profile on  $[0, \infty)$ , and the minimality as above implies strong convergence of  $u_n(0) = u(t_n)$  in  $H^1$ . If  $\xi_{\infty}^0$  is not scattering as  $t \rightarrow -\infty$ , then the same argument for  $t < 0$  implies the same strong convergence.

Suppose that  $\xi_{\infty}^0$  is scattering both as  $t \rightarrow \pm\infty$ . The scattering implies that  $\mathbb{A}_{\varpi}(u_{\infty}^0) \geq 0$ , because  $\mathbb{A}_{\Omega[z]}(\Phi[z]) > 0$  for  $z \in Z_*$ , which follows from

$$\partial_{\omega} \mathbb{A}_{\omega}(\Phi_{\omega}) = \langle \mathbb{A}'_{\omega}(\Phi_{\omega}) | \Phi'_{\omega} \rangle + \mathbb{M}(\Phi_{\omega}) = \mathbb{M}(\Phi_{\omega}) > 0, \quad (6.20)$$

where  $\Phi_{\omega} := \Phi[\Omega|_{[0,z_*]}^{-1}(\omega)]$  is the unique positive radial solution of (2.4).

Moreover, there are non-scattering profiles in  $t > 0$  and in  $t < 0$ . More precisely, there is a profile  $\xi_{\infty}^{l_0}$  with  $s_n^{l_0} \rightarrow \infty$  and  $\|\xi_{\infty}^{l_0}\|_{\text{st}(0,\infty)} = \infty$ , and another profile  $\xi_{\infty}^{l_1}$  with  $-t_n < s_n^{l_1} \rightarrow -\infty$  and  $\|\xi_{\infty}^{l_1}\|_{\text{st}(-\infty,0)} = \infty$ , both satisfying  $\xi_{\infty}^{l_j}(-s_n^{l_j}) = \lambda_n^{l_j}(0) + o(1)$  in  $H^1$ . The last two properties, together with the scattering below the excited states (cf. the argument for (6.6)), imply that, as  $n \rightarrow \infty$ ,

$$(\varpi^{-1/2}\mathbb{H}^0 + \varpi^{1/2}\mathbb{M})(\lambda_n^{l_j}(0)) = \varpi^{-1/2}\mathbb{A}_{\varpi}(u_{\infty}^{l_j}) + o(1) \geq \mathbb{A}^{\varpi}(Q_{\varpi}) + o(1). \quad (6.21)$$

Then the asymptotic orthogonality [5, (7.12)] of the mass-energy implies

$$\begin{aligned} \mathbb{A}^{\varpi}(Q_{\varpi}) + A^* &= \mathbb{A}^{\varpi}(u_{\varpi}) = (\varpi^{-1/2}\mathbb{E} + \varpi^{1/2}\mathbb{M})(u) \\ &\geq \varpi^{-1/2}\mathbb{A}_{\varpi}(u_{\infty}^0) + \sum_{j=0,1} (\varpi^{-1/2}\mathbb{H}^0 + \varpi^{1/2}\mathbb{M})(\lambda_n^{l_j}(0)) + o(1) \\ &\geq 2\mathbb{A}^{\varpi}(Q_{\varpi}) + o(1), \end{aligned} \quad (6.22)$$

which is a contradiction because  $\mathbb{A}^{\varpi}(Q_{\varpi}) \sim 1 \gg A^*$ .

Therefore  $u(t_n) = u_n(0)$  should be strongly convergent (along a subsequence), which means that  $\{u(t)\}_{t \geq 0}$  is precompact. Then the same virial argument as in Section 5.2.1 leads to a contradiction, so the case  $\varpi < \infty$  is precluded. Note that in using the variational lemma 4.1, we can eliminate the cases  $(\mathbb{M} + \omega\mathbb{H}^0)(u_{\omega}) \leq C_M$

and (b) using the scattering by Lemma 2.3 (3) and by [5] respectively (instead of using the proximity to  $Q_\omega$  as in Section 5.2.1).

6.2.2. *The case  $\varpi = \infty$ .* In this case, we apply the profile decomposition of the NLS without potential to the rescaled radiation. Decompose  $u_n = \Phi[z_n] + \eta_n$  and rescale by  $S_n := S_{\omega_n}$ , naming

$$\tilde{z}_n(t) := z_n(t/\omega_n), \quad V_n := V^{\omega_n}, \quad \tilde{\eta}_n := S_n \eta_n(t/\omega_n), \quad \tilde{\Phi}_n := S_n \Phi[\tilde{z}_n]. \quad (6.23)$$

The soliton component is uniformly tending to 0 as

$$|\tilde{z}_n(t)| \lesssim \|u_n(t/\omega_n)\|_2 = \omega_n^{-1/4} \|\tilde{u}_n(t)\|_2 \lesssim \omega_n^{-1/4}. \quad (6.24)$$

The equation for  $\tilde{\eta}_n$  can be written as

$$\begin{aligned} eq_n(\tilde{\eta}_n) &= 2|\tilde{\Phi}_n|^2 \tilde{\eta}_n + \tilde{\Phi}_n^2 \overline{\tilde{\eta}_n} + 2\tilde{\Phi}_n |\tilde{\eta}_n|^2 + \overline{\tilde{\Phi}_n} \tilde{\eta}_n^2 \\ &\quad - \sum_{j=1,2} i\omega_n^{-1} S_n \partial_j \Phi[\tilde{z}_n] \underline{N}_j(\tilde{z}_n, S_n^{-1} \tilde{\eta}_n), \end{aligned} \quad (6.25)$$

where

$$\begin{aligned} eq_n(u) &:= (i\partial_t - \Delta + V_n)u - |u|^2 u, \\ N(z, \eta) &:= 2\Phi[z]|\eta|^2 + \overline{\Phi[z]}\eta^2 + |\eta|^2 \eta, \\ \underline{N}_j(z, \eta) &= \sum_{k=1,2} M_{j,k}^{-1}(z, \eta) \langle N(z, \eta) | \partial_k \Phi[z] \rangle, \\ M_{j,k}(z, \eta) &= \langle i\partial_j \Phi[z] | \partial_k \Phi[z] \rangle - \langle i\eta | \partial_j \partial_k \Phi[z] \rangle, \end{aligned} \quad (6.26)$$

and  $\partial_j \Phi[z]$  denotes the partial derivative in  $(z_1, z_2)$  of  $z = z_1 + iz_2$ . Apply the free profile decomposition to  $\tilde{\eta}_n(0)$  on the time interval  $[0, \infty)$  (see, e.g., [3, Lemma 5.2] or [7, Proposition A.2]):

$$e^{-it\Delta} \tilde{\eta}_n(0) = \sum_{0 \leq j < J} e^{-i(t-s_n^j)\Delta} \lambda^j + \gamma_n^J(t), \quad (6.27)$$

where the sequences of times  $s_n^j \in [0, \infty)$  satisfy

$$s_n^0 \equiv 0, \quad s_n^j - s_n^k \rightarrow \pm\infty \quad (0 \leq j < k < J) \quad (6.28)$$

as  $n \rightarrow \infty$ , and the linear profiles  $\lambda^j \in H_r^1$  are defined by the weak limit

$$\lambda^j = \text{w-} \lim_{n \rightarrow \infty} e^{-is_n^j \Delta} \tilde{\eta}_n(0), \quad (6.29)$$

while the linear remainder  $\gamma_n^J \in C(\mathbb{R}; H_r^1)$  defined by (6.27) satisfies

$$\lim_{J \rightarrow J^*} \limsup_{n \rightarrow \infty} \|\gamma_n^J\|_{[L_t^\infty L^4, \text{Stz}^1]_\theta(0, \infty)} = 0 \quad (6.30)$$

for some  $J^* \in \mathbb{N} \cup \{\infty\}$  and for all  $\theta \in [0, 1)$ , besides the weak vanishing

$$\text{w-} \lim_{n \rightarrow \infty} \gamma_n^J(s_n^j) = 0, \quad (6.31)$$

which follows from (6.29).

Let  $\Lambda^j$  be the nonlinear profile associated with  $\lambda^j$ , and let  $\Gamma_n^J$  be the nonlinear remainder associated with  $\gamma_n^J$ . More precisely, both  $\Lambda^j$  and  $\Gamma_n^J$  are solutions of (1.9) such that

$$\Lambda^0(0) = \lambda^0, \quad \lim_{t \rightarrow -\infty} \|\Lambda^j(t) - e^{-it\Delta} \lambda^j\|_{H^1} = 0 \quad (j \geq 1), \quad \Gamma_n^J(0) = \gamma_n^J(0). \quad (6.32)$$

The small data scattering for (1.9) with (6.30) implies that if  $J$  is close enough to  $J^*$ , then  $\Gamma_n^J$  for large  $n$  is scattering as  $t \rightarrow \infty$  and small in  $[L_t^\infty L^4, \text{Stz}^1]_\theta(0, \infty)$ . In particular, it is small in the weaker norm

$$X := L_t^{16} L^{24/7} \cap L_t^4 L^6 \supset \text{Stz}^{1/2}, \quad (6.33)$$

which is scaling invariant. Fix such  $J < J^*$  for the rest of proof.

Suppose that all the nonlinear profiles  $\Lambda^j$  ( $0 \leq j < J$ ) are scattering as  $t \rightarrow \infty$ , or equivalently  $\Lambda^j \in \text{Stz}^1(0, \infty)$ . Then

$$\tilde{\eta}_n^J := \sum_{0 \leq j < J} \Lambda_n^j + \Gamma_n^J, \quad \Lambda_n^j := \Lambda^j(t + s_n^j), \quad (6.34)$$

is an approximate sequence for  $\tilde{\eta}_n$  in  $X(0, \infty)$ . This claim is based on the long-time perturbation argument together with error estimates on  $eq_n(\tilde{\eta}_n)$  and  $eq_n(\tilde{\eta}_n^J)$ . Note that  $V_n$  in  $eq_n$  is negligible when acting on the given approximate  $\tilde{\eta}_n^J$ , but not on  $\tilde{\eta}_n$  enough to get a closed estimate.  $eq_n(\tilde{\eta}_n)$  and  $eq_n(\tilde{\eta}_n^J)$  are estimated in the norm

$$Y := L_t^4 L^{6/5} + L_t^{8/3} L^{4/3} \quad (6.35)$$

which is a non-admissible dual Strichartz norm, such that we have

$$\left\| \int_0^t e^{i(t-s)(-\Delta + V_n)} P_n f(s) ds \right\|_X \lesssim \|f\|_Y, \quad (6.36)$$

by rescaling [5, Lemma 4.4], where  $P_n$  is the rescaled projection

$$P_n := S_n P_c S_n^{-1}. \quad (6.37)$$

By the same argument as in [5, (7.14)], we deduce from (6.31) that

$$\lim_{n \rightarrow \infty} \|\Gamma_n^J\|_{\text{st}(|t-s_n^j| < \tau)} = 0 \quad (6.38)$$

for any  $\tau \in (0, \infty)$  and any  $0 \leq j < J$ . Combining this, uniform bounds on  $\Lambda^j$  and  $\Gamma_n^J$  in  $\text{Stz}^1 \subset X \cap L_t^4 L^3$ , the fact that the nonlinear part of  $eq_n(\tilde{\eta}_n^J)$  is a linear combination of products of three from  $\{\Lambda_n^j, \Gamma_n^J\}_j$  except the cubic power of each function, and Hölder estimates

$$\begin{aligned} \|fgh\|_{L_t^{8/3} L^{4/3}} &\leq \|f\|_{L_t^{16} L^{24/7}} \|g\|_{L_t^{16} L^{24/7}} \|h\|_{L_t^4 L^6}, \\ \|V_n f\|_{L_t^4 L^{6/5}} &\leq \|V_n\|_{L^2} \|f\|_{L_t^4 L^3} \lesssim \omega_n^{-1/4} \|f\|_{L_t^4 L^3}, \end{aligned} \quad (6.39)$$

using  $V \in L^2$ , we obtain

$$\lim_{n \rightarrow \infty} \|eq_n(\tilde{\eta}_n^J)\|_{Y(0, \infty)} = 0. \quad (6.40)$$

For  $\tilde{\eta}_n$ , the scaling implies

$$\|\tilde{\Phi}_n\|_{L_t^\infty L^3} = \|\Phi_n\|_{L_t^\infty L^3} \sim \|z_n\|_{L_t^\infty} \lesssim \omega_n^{-1/4}, \quad (6.41)$$

hence by Hölder

$$\begin{aligned} \|(\tilde{\Phi}_n)^2 \tilde{\eta}_n\|_{L_t^4 L^{6/5}} &\leq \|\tilde{\Phi}_n\|_{L_t^\infty L^3}^2 \|\tilde{\eta}_n\|_{L_t^4 L^6} \lesssim \omega_n^{-1/2} \|\tilde{\eta}_n\|_X, \\ \|\tilde{\Phi}_n(\tilde{\eta}_n)^2\|_{L_t^{8/3} L^{4/3}} &\leq \|\tilde{\Phi}_n\|_{L_t^\infty L^3} \|\tilde{\eta}_n\|_{L_t^4 L^6}^{3/2} \|\tilde{\eta}_n\|_{L_t^\infty L^3}^{1/2} \lesssim \omega_n^{-1/4} \|\tilde{\eta}_n\|_X^{3/2}. \end{aligned} \quad (6.42)$$

Similarly we have, using  $\|\partial_j \Phi[z]\|_{L^{4/3}} \lesssim 1$ ,

$$\begin{aligned} \|\omega_n^{-1} S_n \partial_j \Phi[\tilde{z}_n] \underline{N}_j(\tilde{z}_n, \eta'_n)\|_{L_t^{8/3} L^{4/3}} &= \|\partial_j \Phi[z_n] \underline{N}_j(z_n, \eta_n)\|_{L_t^{8/3} L^{4/3}} \\ &\lesssim \|\partial_j \Phi[z_n]\|_{L_t^\infty L^{4/3}} \|\Phi[z_n]\| + \|\eta_n\|_{L_t^\infty L^2} \|\eta_n\|_{L_t^4 L^6}^{3/2} \|\eta_n\|_{L_t^\infty L^3}^{1/2} \\ &\lesssim \omega_n^{-1/4} \|\tilde{\eta}_n\|_{L_t^4 L^6}^{3/2} \|\tilde{\eta}_n\|_{L_t^\infty L^3}^{1/2}. \end{aligned} \quad (6.43)$$

Summing these estimates yields

$$\|eq_n(\tilde{\eta}_n)\|_{Y(0,\infty)} \lesssim \omega_n^{-1/2} \|\tilde{\eta}_n\|_X + \omega_n^{-1/4} \|\tilde{\eta}_n\|_X^{3/2}. \quad (6.44)$$

Using the above estimates, and that  $\|\tilde{\eta}_n^J\|_{X(0,\infty)}$  is bounded by the assumption, we see that the error  $e_n^J := \tilde{\eta}_n - \tilde{\eta}_n^J$  satisfies

$$\begin{aligned} &\|(i\partial_t - \Delta + V_n)e_n^J\|_{Y(I)} \\ &\leq \| |\tilde{\eta}_n|^2 \tilde{\eta}_n - |\tilde{\eta}_n^J|^2 \tilde{\eta}_n^J \|_{Y(I)} + \|eq_n(\tilde{\eta}_n)\|_{Y(I)} + \|eq_n(\tilde{\eta}_n^J)\|_{Y(I)} \\ &\lesssim [\|\tilde{\eta}_n^J\|_{X(I)} + \|e_n^J\|_{X(I)} + o(1)]^2 \|e_n^J\|_{X(I)} + o(1), \end{aligned} \quad (6.45)$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ , uniformly on any interval  $I \subset (0, \infty)$ . The global Strichartz estimate works only on the continuous spectrum part  $P_n e_n^J$ , but the other part is under control, because

$$e_n^J = \tilde{\eta}_n - \tilde{\eta}_n^J = R_n P_n e_n^J + (I - R_n P_n) \tilde{\eta}_n^J, \quad (6.46)$$

where  $R_n := S_n R[\tilde{z}_n] S_n^{-1}$ , and so  $1 - R_n P_n = S_n \{1 - R[\tilde{z}_n] P_c\} S_n^{-1}$ , while

$$1 - R[z] P_c = \phi_0 \beta[z] \quad (6.47)$$

for some  $\mathbb{R}$ -linear operator  $\beta[z] : (L^1 + L^\infty)(\mathbb{R}^3) \rightarrow \mathbb{C}$ , which is bounded uniformly for  $z$ . Hence

$$\begin{aligned} \|[1 - R_n P_n] \tilde{\eta}_n^J\|_{L_t^\infty L^3} &\lesssim \|S_n \phi_0\|_{L^3} \|S_n^{-1} \tilde{\eta}_n^J\|_{L_t^\infty L^2} \lesssim \omega_n^{-1/4} \|\tilde{\eta}_n^J\|_{L_t^\infty L^2} = o(1), \\ \|[1 - R_n P_n] \tilde{\eta}_n^J\|_{L_t^4 L^6} &\lesssim \|S_n \phi_0\|_{L^6} \|S_n^{-1} \tilde{\eta}_n^J\|_{L_t^4 L^3} \lesssim \omega_n^{-1/4} \|\tilde{\eta}_n^J\|_{L_t^4 L^3} = o(1), \end{aligned} \quad (6.48)$$

using uniform bounds of  $\tilde{\eta}_n^J$  in  $\text{Stz}^0 \subset L_t^\infty L^2 \cap L_t^4 L^3$ . Since  $L_t^\infty L^3 \cap L_t^4 L^6$  on the left is stronger than  $X$ , and  $R_n(t)$  is uniformly bounded on any  $L^p(\mathbb{R}^3)$ , we deduce that

$$\|e_n^J\|_{X(I)} \lesssim \|P_n e_n^J\|_{X(I)} + o(1), \quad (6.49)$$

as  $n \rightarrow \infty$  uniformly on any interval  $I \subset (0, \infty)$ . For a Gronwall-type argument by the non-admissible Strichartz, it is convenient to introduce the following norm

$$\forall \varphi \in H^1(\mathbb{R}^3), \quad \|\varphi\|_{W_n} := \|e^{it(-\Delta + V_n)} \varphi\|_{X(0,\infty)}. \quad (6.50)$$

Let  $(a, b) \subset (0, \infty)$  such that  $\|\tilde{\eta}_n^J\|_{X(a,b)} \ll 1$ . Then by (6.36), (6.45) and (6.49)

$$\begin{aligned} &\left\{ \|P_n e_n^J\|_{X(a,b)} \right\} \leq \|P_n e_n^J(a)\|_{W_n} + C \|(i\partial_t - \Delta + V_n) e_n\|_{Y(a,b)}, \\ &\|(i\partial_t - \Delta + V_n) e_n\|_{Y(a,b)} \ll \|e_n^J\|_{X(a,b)} + o(1) \lesssim \|P_n e_n^J\|_{X(a,b)} + o(1), \end{aligned} \quad (6.51)$$

where the last term is absorbed by the first, and thus we obtain

$$\max(\|P_n e_n^J(b)\|_{W_n}, \|P_n e_n^J\|_{X(a,b)}) \leq 2\|P_n e_n^J(a)\|_{W_n} + o(1). \quad (6.52)$$

Since  $\tilde{\eta}_n^J$  is bounded in  $X(0, \infty)$ , we can decompose  $(0, \infty)$  into intervals  $I$  of a number  $N$  independent of  $n$  on which the above smallness in  $X(I)$  is valid. Then iterating the above estimate on those intervals, and summing them up, we obtain

$$\|P_n e_n^J\|_{X(0, \infty)} \leq 2^{N+1} \|P_n e_n^J(0)\|_{W_n} + o(1) \lesssim \|e_n^J(0)\|_{H^1} + o(1) = o(1). \quad (6.53)$$

Since it also implies  $\|(i\partial_t - \Delta + V^\omega)e_n^J\|_{Y(0, \infty)} \rightarrow 0$ , the standard Strichartz with Hölder in  $t$  implies that  $\|e_n^J\|_{L_t^\infty L^2(I)} \rightarrow 0$  on any bounded interval  $I$ . Then by interpolation with the  $H^1$  bound, the convergence holds also in  $L_t^\infty H^s(I)$  for all  $s < 1$ . Thus we have proven that if all the nonlinear profiles  $\Lambda^j$  scatter then for any  $s < 1$  and  $T < \infty$ ,

$$\lim_{n \rightarrow \infty} \|\tilde{\eta}_n - \tilde{\eta}_n^J\|_{X(0, \infty) \cap L_t^\infty H^s(0, T)} = 0, \quad \sup_{n \in \mathbb{N}} \|\tilde{\eta}_n^J\|_{X(0, \infty)} < \infty, \quad (6.54)$$

which contradicts that  $\|\tilde{\eta}_n\|_{X(0, \infty)} = \|\eta_n\|_{X(0, \infty)} \sim \|\xi_n\|_{X(0, \infty)} \rightarrow \infty$ . Therefore, at least one profile  $\Lambda^j$  does not scatter. The definition (6.32) implies  $\mathbb{A}(\Lambda^j) = \frac{1}{2}\|\lambda^j\|_{H^1}^2 \geq 0$  for  $j \geq 1$ , then using the asymptotic orthogonality at  $t = 0$ , we have

$$\begin{aligned} \mathbb{A}(\tilde{\eta}_n(0)) &= \mathbb{A}(\Lambda^0) + \sum_{1 \leq j < J} \frac{1}{2} \|\lambda^j\|_{H^1}^2 + \frac{1}{2} \|\gamma_n^J\|_{H^1}^2 + o(1) \\ &= \sum_{0 \leq j < J} \mathbb{A}(\Lambda^j) + \mathbb{A}(\Gamma_n^J) + o(1) \end{aligned} \quad (6.55)$$

as  $n \rightarrow \infty$ . We also have  $\mathbb{E}^0(\Lambda^0) \geq 0$ , since otherwise  $\Lambda^0$  blows up in  $t > 0$ , contradicting that  $\Lambda^0 = \text{w-}\lim_{n \rightarrow \infty} \tilde{\eta}_n$  is bounded in  $H^1$  on  $t \geq 0$ .

Since  $\mathbb{M}(\tilde{u}_n) = \mathbb{M}(\tilde{\Phi}[z_n]) + \mathbb{M}(\tilde{\eta}_n)$ ,  $\|\tilde{\Phi}_n\|_{\dot{H}^1} \lesssim \omega_n^{-1/2}$  and  $|\lceil V_n \rceil(\tilde{u}_n)| \lesssim \omega_n^{-1/4}$  by (2.6), we have

$$\mathbb{A}(\tilde{\eta}_n(0)) \leq \mathbb{A}(\tilde{\eta}_n(0)) + \mathbb{M}(\tilde{\Phi}_n(0)) = \mathbb{A}^{\omega_n}(\tilde{u}_n) + o(1) \leq \mathbb{A}(Q) + A^* + o(1) \quad (6.56)$$

as  $n \rightarrow \infty$ . Since  $A^* \ll \mathbb{A}(Q)$ , we deduce that at most one profile can satisfy  $\mathbb{A}(\Lambda^j) \geq \mathbb{A}(Q)$ , and all the others are below the ground state  $Q$ , and so scattering by [3].

Hence there is exactly one profile  $\Lambda^j$  which is not scattering and  $\mathbb{A}(\Lambda^j) \geq \mathbb{A}(Q)$ . Then the above approximation by  $\tilde{\eta}_n^J$  works up to  $t = s_n^j + O(1)$ , which, together with (6.38), implies that

$$\tilde{\eta}_n(s_n^j + t) \rightarrow \Lambda^j(t) \text{ in w-} H_r^1 \quad (n \rightarrow \infty) \quad (6.57)$$

for any  $t \in \mathbb{R}$  if  $j \geq 1$  and for any  $t \geq 0$  if  $j = 0$ . Hence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{2} \|\tilde{\eta}_n(s_n^j + t) - \Lambda^j(t)\|_{H^1}^2 + \mathbb{M}(\tilde{\Phi}_n(s_n^j + t)) \\ = \limsup_{n \rightarrow \infty} \mathbb{A}(\tilde{\eta}_n(s_n^j + t)) + \mathbb{M}(\tilde{\Phi}_n(s_n^j + t)) - \mathbb{A}(\Lambda^j(t)) \leq A^* \ll \delta_*^2. \end{aligned} \quad (6.58)$$

On the other hand,  $d_{\omega_n}(\tilde{u}_n(s_n^j + t)) \geq \delta_*$ ,  $\|Q_{\omega_n} - Q\|_{H^1} \rightarrow 0$  and  $\|\tilde{\Phi}_n\|_{\dot{H}^1} \rightarrow 0$  imply

$$\|\tilde{\eta}_n(s_n^j + t) - e^{i\theta} Q\|_{H^1} \gtrsim \delta_* - C \|\tilde{\Phi}_n(s_n^j + t)\|_2 + o(1) \gtrsim \delta_*. \quad (6.59)$$

Combining the above two estimates and  $\|\Lambda^j(t) - e^{i\theta}Q\|_{H^1} \ll \delta_*$  yields a contradiction.

Therefore we have a uniform lower bound  $\inf_{\theta \in \mathbb{R}} \|\Lambda^j(t) - e^{i\theta}Q\|_{H^1} \gtrsim \delta_*$ , as well as energy bound  $\mathbb{A}(\Lambda^j) \leq \mathbb{A}(Q) + A^*$ . Hence by the result in [7] for the NLS without potential, if  $A^* \ll \delta_*^2$ , then  $\Lambda^j$  scatters to 0 as  $t \rightarrow \infty$ , which is a contradiction.

Thus we have reached contradiction both for  $\varpi < \infty$  and for  $\varpi = \infty$ . Therefore  $A^* \gtrsim \delta_*^2$  and we have proven

**Lemma 6.2.** *There is a constant  $c_* \in (0, c_X]$  such that for every  $\omega \geq \omega_*$  and every solution  $u_\omega$  of (3.2) satisfying*

$$\mathbb{A}^\omega(u_\omega) \leq \mathbb{A}^\omega(Q_\omega) + c_*\delta_*^2, \quad \inf_{t \geq 0} d_\omega(u_\omega(t)) \geq \delta_*, \quad \mathfrak{S}_\omega(u_\omega(0)) = +1, \quad (6.60)$$

*scatters to  $\Phi$  as  $t \rightarrow \infty$ .*

**6.3. Classification of the dynamics.** Let  $\omega \geq \omega_*$  and let  $u_\omega$  be a solution of (3.2) from  $t = 0$  with the maximal existence time  $T_+ \in (0, \infty]$  satisfying the constraint:

$$\mathbb{A}^\omega(u_\omega) < \mathbb{A}^\omega(Q_\omega) + c_*\delta_*^2, \quad (6.61)$$

where  $\delta_*, c_* > 0$  are the small constants introduced in Lemmas 5.2 and 6.2.  $u(t) := S_\omega^{-1}u_\omega(\omega t)$  solves the original equation (1.1) on  $[0, T_+/\omega]$  with

$$\mathbb{A}_\omega(u) < \mathbb{A}_\omega(\Psi[\omega]) + \omega^{1/2}c_*\delta_*^2. \quad (6.62)$$

The distance function in the rescaled variable is abbreviated as before by

$$d(t) := d_\omega(u_\omega(t)). \quad (6.63)$$

If  $\inf_{0 \leq t < T_+} d(t) \geq \delta_*$ , then (6.61) and  $c_* \leq c_X$  imply that  $u_\omega(t) \in \check{\mathcal{H}}_\omega$  for all  $t \in [0, T_+)$ . Moreover, Lemmas 6.1 and 6.2 imply

$$\mathfrak{S}_\omega(u_\omega(0)) = \begin{cases} +1 & \implies u_\omega \text{ scatters to } \Phi \text{ as } t \rightarrow \infty, \\ -1 & \implies u_\omega \text{ blows up in } t > 0. \end{cases} \quad (6.64)$$

If  $\inf_{0 \leq t < T_+} d(t) < \delta_*$ , then the one-pass lemma 5.2 implies that  $d(t) < \delta_*$  on  $t \in (t_1, t_2)$  and  $d(t) > \delta_*$  on  $t \in (t_2, T_+)$ , for  $t_1, t_2 \in [0, T_+]$  defined by

$$t_1 := \inf\{t \in [0, T_+) \mid d(t) < \delta_*\}, \quad t_2 := \sup\{t \in [0, T_+) \mid d(t) < \delta_*\}. \quad (6.65)$$

If  $t_1 > 0$ , then applying the ejection lemma 3.3 from  $t = t_1$  backward, there exists  $t_0 < t_1$  such that  $d(t)$  is strictly and exponentially decreasing on  $[t_0, t_1]$  with

$$d(t) \sim e^{-\alpha_\omega(t-t_0)}\delta_X \sim e^{-\alpha_\omega(t_1-t)}\delta_*, \quad d(t_0) = \delta_X > d(t_1) = \delta_*. \quad (6.66)$$

If  $t_2 < T_+$ , then we have the same dichotomy as in (6.64) at  $t = t_2$ .

If  $t_2 = T_+$ , then the uniform bound  $d(t) \leq \delta_*$  for  $t \geq t_1$  implies  $t_2 = T_+ = \infty$ , and by the trapping lemma 3.5, there exists  $t_3 \in [t_1, \infty]$  such that  $d(t)$  is strictly and exponentially decreasing on  $[t_1, t_3]$  with

$$\begin{cases} t_1 \leq t < t_3 & \implies d(t) \sim e^{-\alpha_\omega(t-t_1)}\delta_*, \\ t_3 < t < \infty & \implies c_X d(t)^2 < \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega). \end{cases} \quad (6.67)$$

This implies that  $u(t) \in \mathcal{N}_\delta$  for large  $t$  and for some  $\delta \sim (c_*/c_X)^{1/2}\delta_*$ , so  $u$  is trapped by  $\Psi$  as  $t \rightarrow \infty$ . We have  $t_3 = \infty$  if and only if  $u(t)$  is strongly convergent to  $e^{-i\omega(t-a)}\Psi[\omega]$  in  $H_r^1$  as  $t \rightarrow \infty$  for some  $a \in \mathbb{R}$ .

Now that we have proven the classification part of the main Theorem 1.1, together with some description of each behavior, it remains to see for which initial data each of the possibilities occurs, especially for the trapping and the transition.

## 7. CENTER-STABLE MANIFOLD OF THE EXCITED SOLITONS

In this section, we show that the set of initial data for which the solution is trapped by  $\Psi$  is a  $C^1$  manifold of codimension 1, and that it is a threshold between the scattering to  $\Phi$  and the blow-up. It is a center-stable manifold of  $\mathcal{S}_1|_{\mathbb{M} \ll 1}$ , its time inversion is a center-unstable manifold, and there are all the 9 types of solutions around the transversal intersection of them.

**7.1. Construction around the excited states.** First we construct a manifold around a fixed excited soliton  $e^{-it}Q_\omega$  by the bisection argument as a graph of  $(b_-(0), \zeta(0)) \mapsto b_+(0)$  in the decomposition (2.71).

**Theorem 7.1.** *Take  $\delta_\pm \in (0, \delta_X)$  such that  $\delta_-/\delta_+, \delta_+/\delta_X$  and  $(\delta_- \delta_+ + \delta_+^3)/(c_X \delta_X^2)$  are all small enough. Then a unique  $C^1$  function  $G_\omega$  is defined for each  $\omega \geq \omega_*$  on*

$$U_\omega := \{(b_-, \zeta) \in \mathbb{R} \times \mathcal{Z}_\omega \mid \max(|b_-|, \|\zeta\|_\omega) < \delta_-\}, \quad (7.1)$$

such that for any

$$(\theta, b_+, b_-, \zeta) \in \mathcal{U}_\omega := (\mathbb{R}/2\pi\mathbb{Z}) \times (-\delta_+, \delta_+) \times U_\omega, \quad (7.2)$$

the solution of (3.2) with the initial condition  $u_\omega(0) = \mathcal{C}_\omega(\theta, b_+, b_-, \zeta)$  satisfies

- (1) If  $b_+ = G_\omega(b_-, \zeta)$  then  $d_\omega(u_\omega(t)) < \delta_X/2$  for all  $t \geq 0$ .
- (2) If  $b_+ \neq G_\omega(b_-, \zeta)$  then  $d_\omega(u_\omega(t))$  reaches  $\delta_X$  at some  $t_X > 0$ , where

$$\mathcal{B}_1^\omega(u_\omega(t_X)) \sim \text{sign}(b_+ - G_\omega(b_-, \zeta))\delta_X. \quad (7.3)$$

Moreover, we have  $|G_\omega(b_-, \zeta)| \lesssim |b_-|^2 + \|\zeta\|_\omega^2$ .

The above characterization (1)-(2) implies that the value of  $G_\omega$  is independent of the choice of  $\delta_\pm$ . The distance upper bound  $\delta_X/2$  in the case (1) is chosen just for distinction from the case (2), but it can be made arbitrarily small by taking  $\delta_\pm$  smaller.

**7.1.1. Existence of  $G_\omega$ .** First, we prove the existence of a value of  $b_+$  for which  $u_\omega$  is trapped. Fix  $\delta_\pm$  such that  $0 < \delta_- \ll \delta_+ \ll \delta_X$  and  $\delta_- \delta_+ + \delta_+^3 \ll c_X \delta_X^2$ . Take any  $(b_-, \zeta) \in U_\omega$ . Since  $d_\omega(u_\omega(0)) \lesssim \delta_+ \ll \delta_X$ , if  $d_\omega(u_\omega(t))$  reaches  $\delta_X$  at some  $t = t_X > 0$ , then the ejection lemma 3.3 implies  $|b_1(t_X)| \sim \delta_X$ . Let  $B_\pm$  be the sets of such  $b_+ \in (-\delta_+, \delta_+)$  that  $b_1(t_X) \sim \pm \delta_X$  at the first ejection time in  $t > 0$ .

$B_\pm$  are open, because the ejection lemma applies to perturbed solutions  $u'_\omega$  as long as  $d_\omega(u'_\omega(t_X)) \sim \delta_X \gg d_\omega(u'_\omega(0))$ , while  $\text{sign } b_1$  remains constant.  $B_+ \cap B_- = \emptyset$  by definition. Both sets are non-empty, because for  $|b_+| \gg |b_-| + \|\zeta\|_\omega$ , (2.74) and (3.25) imply that the ejection condition (3.26) is satisfied at  $t = 0$ , then  $\text{sign } b_1(t) = \text{sign } b_+(t)$  is preserved until  $d_\omega(u_\omega(t))$  reaches  $\delta_X$ .

Hence by the connectedness,  $(-\delta_+, \delta_+) \setminus (B_+ \cup B_-)$  is not empty either. If  $b_+$  is in this set, then by definition of  $B_\pm$ , we have  $d_\omega(u_\omega(t)) < \delta_X$  for all  $t \geq 0$ , and so



the trapping lemma 3.5 applies to  $u_\omega$  on  $t \geq 0$ . Since

$$\begin{aligned} \mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega) &= -2b_+b_- + \frac{1}{2}\langle \mathcal{L}^\omega \zeta | \zeta \rangle - C^\omega(b_+g_+^\omega + b_-g_-^\omega + \zeta) \\ &\lesssim \delta_- \delta_+ + \delta_+^3, \end{aligned} \quad (7.4)$$

the trapping lemma implies that for all  $t \geq 0$

$$\begin{aligned} d_\omega(u_\omega(t))^2 &\leq \min(d_\omega(u_\omega(0))^2, c_X^{-1}(\mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega))) \\ &\lesssim \min(\delta_+^2, c_X^{-1}(\delta_- \delta_+ + \delta_+^3)) \ll \delta_X^2. \end{aligned} \quad (7.5)$$

**7.1.2. Lipschitz estimate.** Next we prove a key Lipschitz estimate for a generalized difference equation of (3.12) for trapped solutions, which will imply the uniqueness and Lipschitz continuity of  $G_\omega$ .

Before taking the difference, we prepare time-local bound on the Strichartz norm. Let  $v$  be a solution of (3.12) on an interval  $I$ . Applying the Strichartz estimate of  $e^{-it\Delta}$  to the equation of  $v$ , we deduce that there exists a small constant  $\delta_S \in (0, 1)$  such that

$$\|v\|_{L_t^\infty H^1(I)} \leq \delta_S \quad \text{and} \quad |I| \leq 1 \implies \|v\|_{\text{Stz}^1(I)} \lesssim \|v\|_{L_t^\infty H^1(I)}. \quad (7.6)$$

In particular, denoting

$$\|u\|_{\text{Stz}_{\text{UL}}^1(I)} := \sup_{J \subset I, |J| \leq 1} \|u\|_{\text{Stz}^1(J)}, \quad (7.7)$$

we have

$$\|v\|_{L_t^\infty H^1(0, \infty)} \leq \delta_S \implies \|v\|_{\text{Stz}_{\text{UL}}^1(0, \infty)} \lesssim \|v\|_{L_t^\infty H^1(0, \infty)}. \quad (7.8)$$

Now let  $v^0, v^1$  be two solutions of (3.12), and let  $\vec{v} := (v^0, v^1)$ . Then the difference  $\triangleleft v^\triangleright = v^1 - v^0$  satisfies

$$(\partial_t - i\mathcal{L}^\omega)\triangleleft v^\triangleright = -iQ_\omega \triangleleft m^\omega(v^\triangleright) + \triangleleft \mathcal{N}^\omega(v^\triangleright) = [-iQ_\omega \acute{m}^\omega(\vec{v}) + \acute{\mathcal{N}}^\omega(\vec{v})]\triangleleft v^\triangleright, \quad (7.9)$$

where  $\acute{m}^\omega(\vec{v})$  and  $\acute{\mathcal{N}}^\omega(\vec{v})$  are operators defined by the following: for any function  $X(v)$  which is Fréchet differentiable in  $v$ , and for  $\vec{v} = (v^0, v^1)$ , define

$$\acute{X}(\vec{v}) := \int_0^1 X'((1-\theta)v^0 + \theta v^1) d\theta, \quad (7.10)$$

so that the difference of  $X$  at  $v^0$  and  $v^1$  can be written as

$$\triangleleft X(v^\triangleright) = \acute{X}(\vec{v})\triangleleft v^\triangleright. \quad (7.11)$$

The Fréchet derivatives  $(N^\omega)'(v), (\mathcal{N}^\omega)'(v) : H^1 \rightarrow H^{-1}$  and  $(m^\omega)'(v) : H^1 \rightarrow \mathbb{R}$  can be written explicitly as follows.

$$\begin{aligned} (N^\omega)'(v)\varphi &= 2Q_\omega(3v_1\varphi_1 + v_2\varphi_2) + 3v_1^2\varphi_1 + 2v_1v_2\varphi_2 \\ &\quad + i[2Q_\omega(v_2\varphi_1 + v_1\varphi_2) + 2v_1v_2\varphi_1 + 3v_2^2\varphi_2], \\ (m^\omega)'(v)\varphi &= [\langle Q_\omega | Q'_\omega \rangle + \langle v | Q'_\omega \rangle]^{-2} [\langle v | Q_\omega \rangle + \langle N^\omega(v) | Q'_\omega \rangle] \langle Q'_\omega | \varphi \rangle \\ &\quad - [\langle Q_\omega | Q'_\omega \rangle + \langle v | Q'_\omega \rangle]^{-1} [\langle Q_\omega | \varphi \rangle + \langle (N^\omega)'(v) \varphi | Q'_\omega \rangle], \\ (i\mathcal{N}^\omega)'(v)\varphi &= [(m^\omega)'(v)\varphi]v + m^\omega(v)\varphi + (N^\omega)'(v)\varphi. \end{aligned} \quad (7.12)$$

A similar expression is obtained for  $\dot{N}^\omega(\vec{v})$  by replacing  $v$  in  $(N^\omega)'(v)$  with  $v^\natural$  and  $v^\perp$ , taking the linear combination of such terms. The computation for  $\dot{m}^\omega(\vec{v})$  is slightly more complicated because of the quotient, but still elementary.

The above equation (7.9) is linear in  $\langle v^\triangleright$ , so the difference quotient, as well as its limit, namely the derivative, solves the same form of equation. Hence it is convenient to derive a Lipschitz estimate for general solutions  $v^\diamond$  of the linear equation

$$(\partial_t - i\mathcal{L}^\omega)v^\diamond = [-iQ_\omega \dot{m}^\omega(\vec{v}) + \dot{\mathcal{N}}^\omega(\vec{v})]v^\diamond, \quad \vec{v} := (v^\natural, v^\perp), \quad (7.13)$$

where  $v^\natural, v^\perp$  are given functions satisfying for some small  $\delta > 0$ ,

$$\max_{j=0,1} \|v^j\|_{\text{Stz}_{\text{UL}}^1(0,\infty)} \leq \delta. \quad (7.14)$$

In other words, we ignore the relation  $\langle v^\triangleright = v^\perp - v^\natural$  in (7.9).

It is easy to see, using the Strichartz estimate, that (7.13) is wellposed for  $H^1 \ni v^\diamond(0) \mapsto v^\diamond \in \text{Stz}_{\text{loc}}^1(0,\infty)$ , and that the solution satisfies

$$\partial_t \langle iv^\diamond | Q'_\omega \rangle = 0 \quad (7.15)$$

(by differentiating the equation (3.10) of  $m^\omega$ ). Hence the orthogonality  $\langle iv^\diamond | Q'_\omega \rangle = 0$  is preserved if it is initially fulfilled.

$v^\diamond$  is decomposed as before by the symplectic orthogonality

$$v^\diamond = b_+^\diamond g_+^\omega + v_{\text{cs}}^\diamond = b_+^\diamond g_+^\omega + b_-^\diamond g_-^\omega + \zeta^\diamond, \quad b_\pm^\diamond := P_\pm^\omega v^\diamond, \quad \langle i\zeta^\diamond | g_\pm^\omega \rangle = 0. \quad (7.16)$$

Then using the equation (7.13), we obtain

$$\begin{aligned} |(\partial_t \mp 2\alpha_\omega)|b_\pm^\diamond|^2| &= 2|\langle P_\pm^\omega \dot{\mathcal{N}}^\omega(\vec{v})v^\diamond | b_\pm^\diamond \rangle| \\ &\lesssim \|\vec{v}\|_{H^1} \|v^\diamond\|_{H^1} |b_\pm^\diamond| \lesssim \delta \|v^\diamond\|_{H^1} |b_\pm^\diamond|, \end{aligned} \quad (7.17)$$

and, using Hölder and partial integration in  $x$ ,

$$\begin{aligned} |\partial_t \|\zeta^\diamond\|_\omega^2| &= |\langle P_\epsilon^\omega \dot{\mathcal{N}}^\omega(\vec{v})v^\diamond | \mathcal{L}^\omega \zeta^\diamond \rangle| \\ &\lesssim [\|\vec{v}\|_{L^\infty} + \|\vec{v}\|_{L^\infty}^2] \|v^\diamond\|_{H^1} \|\zeta^\diamond\|_{H^1} \\ &\quad + [\|\nabla \vec{v}\|_{L^3} + \|\vec{v}\|_{L^\infty} \|\nabla \vec{v}\|_{L^3}] \|v^\diamond\|_{L^6} \|\zeta^\diamond\|_{H^1}. \end{aligned} \quad (7.18)$$

Hence for any interval  $I \subset [0, \infty)$  with length  $|I| \leq 1$ ,

$$[\|\zeta^\diamond\|_\omega^2]_{\partial I} \lesssim \delta \|v^\diamond\|_{L_t^\infty H^1(I)} \|\zeta^\diamond\|_{L_t^\infty H^1(I)}, \quad (7.19)$$

using that  $\|\vec{v}\|_{L_t^4(W^{1,3} \cap L^\infty)(I)} \lesssim \|\vec{v}\|_{\text{Stz}^1(I)}$ .

Combining (7.17) and (7.19), we deduce that there exist absolute constants  $C_0 \in (1, \infty)$  and  $\delta_0 \in (0, 1)$  such that if  $\delta \leq \delta_0$  then for every  $t_0 \geq 0$

$$\begin{cases} \sup_{t_0 \leq t \leq t_0+1} \|v^\diamond\|_\omega \leq C_0 \|v^\diamond(t_0)\|_\omega, \\ \sup_{t_0 \leq t \leq t_0+1} \|v_{\text{cs}}^\diamond\|_\omega \leq \|v_{\text{cs}}^\diamond(t_0)\|_\omega + C_0 \delta \|v^\diamond(t_0)\|_\omega. \end{cases} \quad (7.20)$$

Suppose that for some  $\ell > 0$  and  $t_0 \geq 0$ , we have

$$\|v_{\text{cs}}^\diamond(t_0)\|_\omega \leq \ell |b_+^\diamond(t_0)|, \quad (7.21)$$

and define  $t_1 \in (t_0, \infty]$  by

$$t_1 = \inf\{t > t_0 \mid \|v_{\text{cs}}^\diamond(t)\|_\omega > (\ell + C_0\delta + C_0\delta\ell)|b_+^\diamond(t)|\}. \quad (7.22)$$

If  $\delta, \ell > 0$  are chosen such that

$$\delta(\ell + C_0\delta + C_0\delta\ell) \ll \alpha, \quad (7.23)$$

then for  $t \in [t_0, t_1)$  we have, using  $\alpha_\omega \in (\frac{9}{10}\alpha, \frac{11}{10}\alpha)$ ,

$$\delta\|v^\diamond\|_\omega \ll \alpha_\omega|b_+^\diamond|, \quad (7.24)$$

and injecting this into (7.17),

$$\partial_t|b_+^\diamond|^2 \geq \alpha_\omega|b_+^\diamond|^2. \quad (7.25)$$

Hence  $|b_+^\diamond(t)|$  is increasing on  $[t_0, t_1)$ . On the other hand, (7.20) implies

$$\begin{aligned} t_0 \leq t \leq t_0 + 1 &\implies \|v_{\text{cs}}^\diamond\|_\omega \leq (1 + C_0\delta)\|v_{\text{cs}}^\diamond(t_0)\|_\omega + C_0\delta|b_+^\diamond(t_0)| \\ &\leq (\ell + C_0\delta + C_0\delta\ell)|b_+^\diamond(t_0)|. \end{aligned} \quad (7.26)$$

Therefore by the definition of  $t_1$ , we deduce that

$$t_1 > t_0 + 1. \quad (7.27)$$

In particular, we obtain from (7.25) and (7.26),

$$\|v_{\text{cs}}^\diamond(t_0 + 1)\|_\omega \leq (\ell + C_0\delta + C_0\delta\ell)e^{-\alpha_\omega/2}|b_+^\diamond(t_0 + 1)|. \quad (7.28)$$

Then imposing another condition on  $(\delta, \ell)$ :

$$(\ell + C_0\delta + C_0\delta\ell) \leq \ell e^{\alpha/3} \quad (7.29)$$

leads to

$$\|v_{\text{cs}}^\diamond(t_0 + 1)\|_\omega \leq \ell|b_+^\diamond(t_0 + 1)|, \quad (7.30)$$

so by induction we deduce that for all  $n \in \mathbb{N}$ ,

$$\|v_{\text{cs}}^\diamond(t_0 + n)\|_\omega \leq \ell|b_+^\diamond(t_0 + n)|. \quad (7.31)$$

Moreover,  $t_1 = \infty$  and (7.25) is valid for all  $t \geq t_0$ . Thus we have obtained the following key lemma, choosing  $C_1 \gg C_0$ .

**Lemma 7.2.** *There is a constant  $C_1 \in (2C_0, \infty)$  such that if  $\delta > 0$  is small enough and  $\omega \geq \omega_*$ ,  $v^j$  satisfies  $\|v^j\|_{\text{Stz}_{\text{UL}}^1(0, \infty)} \leq \delta$  for  $j = 0, 1$ , and  $v^\diamond \in C([0, \infty); H^1)$  satisfies the equation (7.13) for  $t \geq 0$ , together with  $\langle iv^\diamond(0) | Q'_\omega \rangle = 0$  and*

$$\|P_{\text{cs}}^\omega v^\diamond(0)\|_\omega \leq \ell|P_+^\omega v^\diamond(0)| \quad (7.32)$$

for some  $\ell > 0$  in the range

$$\frac{C_1\delta}{\alpha} \leq \ell \leq \frac{\alpha}{C_1\delta}, \quad (7.33)$$

then for all  $n \in \mathbb{N}$  and all  $t \geq 0$  we have

$$\|P_{\text{cs}}^\omega v^\diamond(n)\|_\omega \leq \ell|P_+^\omega v^\diamond(n)|, \quad \|P_{\text{cs}}^\omega v^\diamond(t)\|_\omega \leq \ell(1 + C_1\delta)|P_+^\omega v^\diamond(t)|, \quad (7.34)$$

and

$$|P_+^\omega v^\diamond(t)| \geq e^{\alpha_\omega t/2}|P_+^\omega v^\diamond(0)|. \quad (7.35)$$

Note that (7.33) is a sufficient condition to have (7.23) and (7.29), and the range of  $\ell$  is non-empty for  $0 < \delta \leq \alpha/C_1$ .

7.1.3. *Uniqueness and regularity of  $G_\omega$ .* Let  $0 < \delta \ll 1$  and  $\ell > 0$  satisfy (7.33). Let  $v^0, v^1$  be two solutions of (3.12) satisfying  $v^i(0) \in \mathcal{V}_\omega$  and  $\|\vec{v}\|_{\text{Stz}_{\text{UL}}^1(0, \infty)} \leq \delta$ . Then  $\langle v^\triangleright = v^1 - v^0$  satisfies the equation (7.13). Suppose that at some  $t_0 \geq 0$  we have

$$\|\langle v_{\text{cs}}^\triangleright(t_0)\|_\omega \leq \ell |\langle b_+^\triangleright(t_0)||. \quad (7.36)$$

Then the above lemma implies that  $\langle b_+^\triangleright$  is exponentially growing for  $t \geq t_0$ , which contradicts  $\vec{v} \in L_t^\infty H^1(0, \infty)$ , unless  $\langle v^\triangleright(t_0) = 0$ . Hence for all  $t \geq 0$ , we have

$$|\langle b_+^\triangleright(t)|| \leq \ell^{-1} \|\langle v_{\text{cs}}^\triangleright(t)\|_\omega, \quad (7.37)$$

where the Lipschitz constant can be optimized by taking the largest possible  $\ell = O(\delta^{-1})$  in the lemma. Then going back to (7.20), we also obtain

$$\|\langle v_{\text{cs}}^\triangleright(t)\|_\omega \leq e^{2C_0\delta(t+1)} \|\langle v_{\text{cs}}^\triangleright(0)\|_\omega \quad (7.38)$$

for all  $t \geq 0$ . Thus we obtain (using  $C_1 \geq 2C_0$ )

**Lemma 7.3.** *Let  $0 < \delta < \alpha/C_1$  be small enough and  $\omega \geq \omega_*$ . Let  $v^0, v^1$  be two solutions of (3.12) on  $t \in [0, \infty)$  satisfying the orthogonality  $v^i(0) \in \mathcal{V}_\omega$  and  $\|v^i\|_{L_t^\infty H^1(0, \infty)} \leq \delta$ . Then we have, for all  $t \geq 0$ ,*

$$\alpha |\langle P_+^\omega v^\triangleright(t)|| \leq C_1 \delta \|\langle P_{\text{cs}}^\omega v^\triangleright(t)\|_\omega, \quad (7.39)$$

and for all  $t \geq 0$ ,

$$\|\langle P_{\text{cs}}^\omega v^\triangleright(t)\|_\omega \leq e^{C_1\delta(t+1)} \|\langle P_{\text{cs}}^\omega v^\triangleright(0)\|_\omega. \quad (7.40)$$

The above lemma implies the uniqueness of  $G_\omega(b_-, \zeta)$  for each small  $(b_-, \zeta)$ , as well as the Lipschitz continuity. To show the Gâteaux differentiability, fix arbitrary  $\varphi, \psi \in \mathcal{Z}_\omega$  and  $a, b \in \mathbb{R}$  such that  $\|\varphi\|_\omega + |a| \ll 1$ , and let  $v^0, v^1$  be two solutions of (3.12) satisfying  $\|v^i\|_{L_t^\infty H^1(0, \infty)} \leq \delta$  and

$$P_{\text{cs}}^\omega v^0(0) = ag_-^\omega + \varphi, \quad P_{\text{cs}}^\omega v^1(0) = (a + hb)g_-^\omega + (\varphi + h\psi) \quad (7.41)$$

with a small parameter  $\mathbb{R} \ni h \rightarrow 0$ .

Then  $w := \langle v^\triangleright/h$  solves the equation (7.13) with the initial condition  $P_{\text{cs}}^\omega w(0) = bg_-^\omega + \psi$  independent of  $h$ , and the above lemma implies that for all  $t \geq 0$

$$\alpha |P_+^\omega w(t)| \leq C_1 \delta \|P_{\text{cs}}^\omega w(t)\|_\omega, \quad \|P_{\text{cs}}^\omega w(t)\|_\omega \leq e^{C_1\delta(t+1)} \|P_{\text{cs}}^\omega w(0)\|_\omega. \quad (7.42)$$

Using the local wellposedness of (7.13) as well, we deduce that  $w$  is bounded in  $\text{Stz}^1(0, T)$  as  $h \rightarrow 0$  for any  $T < \infty$ . The uniform bound together with the equation implies that there is a sequence of  $h \rightarrow 0$  along which  $w$  converges to some  $w_\infty \in \text{Stz}_{\text{loc}}^1(0, \infty)$  in  $C([0, \infty); w\text{-}H^1)$ . The limit  $w_\infty$  solves the equation (7.13) with  $\vec{v} = (v, v)$ , satisfying

$$\alpha |P_+^\omega w_\infty(t)| \leq C_1 \delta \|P_{\text{cs}}^\omega w_\infty(t)\|_\omega, \quad \|P_{\text{cs}}^\omega w_\infty(t)\|_\omega \leq e^{C_1\delta(t+1)} \|P_{\text{cs}}^\omega w_\infty(0)\|_\omega, \quad (7.43)$$

where the last normand is the prescribed  $bg_-^\omega + \psi$ . If there is another limit  $w'_\infty$  along another sequence of  $h \rightarrow 0$ , then  $w_\infty - w'_\infty$  satisfies the same equation (7.13) and the same estimates with  $P_{\text{cs}}^\omega(w_\infty - w'_\infty)(0) = 0$ , therefore  $w_\infty \equiv w'_\infty$ . Hence the limit is unique, and so the convergence holds for the entire limit  $h \rightarrow 0$ . Thus we obtain the Gâteaux derivative of  $G_\omega$  at  $(a, \varphi)$  in the direction  $(b, \psi)$

$$G'_\omega(a, \varphi)(b, \psi) = P_+^\omega w_\infty(0) \in \mathbb{R}, \quad (7.44)$$

which is bounded linear on  $(b, \psi) \in \mathbb{R} \times \mathcal{Z}_\omega$ , because it is determined by the linear equation (7.13) with the boundedness by (7.43):

$$\|G'_\omega(a, \varphi)\|_{\mathcal{B}(\mathbb{R} \times \mathcal{Z}_\omega, \mathbb{R})} \lesssim C_1 \alpha^{-1} \|(a, \varphi)\|_{\mathbb{R} \times H^1}. \quad (7.45)$$

To show that the Gâteaux derivative is continuous with respect to  $(a, \varphi)$  in the operator norm, take any sequence  $(a_n, \varphi_n) \in \mathbb{R} \times \mathcal{Z}_\omega$  strongly convergent to  $(a, \varphi)$ , and any sequence  $(b_n, \psi_n) \in \mathbb{R} \times \mathcal{Z}_\omega$  weakly convergent to  $(b, \psi)$ . Let  $v_n$  be the solution of (3.12), and  $w_n$  be the solution of (7.13) with  $\vec{v} = (v_n, v_n)$ , satisfying

$$\begin{aligned} v_n(0) &= G_\omega(a_n, \varphi_n)g_+^\omega + a_n g_-^\omega + \varphi_n, \\ w_n(0) &= G'_\omega(a_n, \varphi_n)(b_n, \psi_n)g_+^\omega + b_n g_-^\omega + \psi_n. \end{aligned} \quad (7.46)$$

By the local wellposedness for (3.12), we have  $v_n \rightarrow v_\infty$  in  $\text{Stz}_{\text{loc}}^1(0, \infty)$ , where  $v_\infty$  is the solution of (3.12) with

$$v_\infty(0) = G_\omega(a, \varphi)g_+^\omega + a g_-^\omega + \varphi. \quad (7.47)$$

Also we have

$$\|w_n(t)\|_\omega \lesssim e^{C_1 \delta t} [|b_n| + \|\psi_n\|_{H^1}], \quad (7.48)$$

which is uniformly bounded on any finite interval. These uniform bounds together with the equation for  $w_n$  imply that  $w_n$  converges to some  $w \in \text{Stz}_{\text{loc}}^1(0, \infty)$  in  $C([0, \infty); w-H^1)$ , at least along a subsequence. Then the limit  $w_\infty$  solves (7.13) with  $\vec{v} = (v_\infty, v_\infty)$  and  $P_{\text{cs}}^\omega w_\infty(0) = b g_-^\omega + \psi$ , satisfying the orthogonality and

$$\max \left[ \frac{\alpha}{C_1 \delta} |P_+^\omega w_\infty(t)|, \|P_{\text{cs}}^\omega w_\infty(t)\|_\omega \right] \leq e^{C_1 \delta(t+1)} \|P_{\text{cs}}^\omega w_\infty(0)\|_\omega. \quad (7.49)$$

The uniqueness of such a solution implies

$$P_+^\omega w_\infty(0) = G'_\omega(a, \varphi)(b, \psi) \quad (7.50)$$

as well as the convergence of  $w_n$  along the full sequence  $n \rightarrow \infty$ . Hence

$$\lim_{n \rightarrow \infty} G'_\omega(a_n, \varphi_n)(b_n, \psi_n) = G'_\omega(a, \varphi)(b, \psi). \quad (7.51)$$

Since this holds for any weakly convergent  $(b_n, \psi_n)$ , we have the convergence of  $G'_\omega(a_n, \varphi_n) \rightarrow G'_\omega(a, \varphi)$  in the operator norm. In short,  $G_\omega$  is a  $C^1$  function on a small neighborhood of 0 in  $\mathbb{R} \times \mathcal{Z}_\omega$ .

**7.2. Nine sets of solutions around the excited states.** We have obtained a manifold for each  $\omega \geq \omega_*$  in the local coordinate  $\mathcal{C}_\omega$  of Lemma 2.6, that is

$$\begin{aligned} \mathcal{M}_0^\omega &:= \{\mathcal{M}_\omega(\theta, b_-, \zeta) \mid (b_-, \zeta) \in U_\omega, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}, \\ \mathcal{M}_\omega(\theta, b_-, \zeta) &:= \mathcal{C}_\omega(\theta, G_\omega(b_-, \zeta), b_-, \zeta), \end{aligned} \quad (7.52)$$

consisting of trapped solutions, in the open neighborhood  $\mathcal{C}_\omega(\mathcal{U}_\omega)$  of  $\mathcal{Q}_\omega$ .  $\mathcal{M}_0^\omega$  has codimension 1, separating the complement into two open sets

$$\mathcal{M}_\pm^\omega := \mathcal{C}_\omega(\{(\theta, b_+, b_-, \zeta) \in \mathcal{U}_\omega \mid \pm(b_+ - G_\omega(b_-, \zeta)) > 0\}). \quad (7.53)$$

The solutions in  $\mathcal{M}_\pm^\omega$  are ejected with  $b_1(t_X) \sim \pm \delta_X$  at some ejection time  $t_X > 0$  (which depends on the solution).

The time inversion of  $\mathcal{M}_0^\omega$  is the complex conjugate

$$\overline{\mathcal{M}_0^\omega} = \{\mathcal{C}_\omega(\theta, b_+, G_\omega(b_+, \bar{\zeta}), \zeta) \mid (b_+, \zeta) \in U_\omega, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}, \quad (7.54)$$

which is a  $C^1$  manifold of codimension 1 consisting of solutions trapped by  $\mathcal{Q}_\omega$  for  $t \leq 0$ . Since  $\overline{g_\pm^\omega} = g_\mp^\omega$  and  $|G'_\omega| \ll 1$ ,  $\mathcal{M}_0^\omega$  and  $\overline{\mathcal{M}_0^\omega}$  intersect transversely: the implicit function theorem yields a unique  $C^1$  function  $\vec{G}_\omega : \{\zeta \in \mathcal{Z}_\omega \mid \|\zeta\|_\omega < \delta_-\} \rightarrow \{(b_+, b_-) \in \mathbb{R}^2 \mid \max|b_\pm| < \delta_-\}$  such that

$$(b_+, b_-) = \vec{G}_\omega(\zeta) \iff b_+ = G_\omega(b_-, \zeta) \text{ and } b_- = G_\omega(b_+, \bar{\zeta}), \quad (7.55)$$

and  $\mathcal{U}_\omega \setminus (\mathcal{M}_0^\omega \cap \overline{\mathcal{M}_0^\omega})$  consists of four open sets with distinct local behavior in  $t > 0$  and in  $t < 0$ , according to  $\text{sign}(b_+ - G_\omega(b_-, \zeta))$  and  $\text{sign}(b_- - G_\omega(b_+, \bar{\zeta}))$ . Thus all solutions of (3.2) starting near  $\mathcal{Q}_\omega$  are classified into 9 non-empty sets of solutions:

$$\{\mathcal{M}_j^\omega \cap \overline{\mathcal{M}_k^\omega}\}_{j,k \in \{0, \pm\}}. \quad (7.56)$$

All these about the manifolds rely only on the instability, and they are independent of the global dynamics investigated in the previous sections.

Under the constraints (6.61) and  $\omega \geq \omega_*$ , the scattering/blow-up away from the excited states also applies to them, leading to the characterization by global behavior:

$$\mathcal{M}_+^\omega \cap \mathcal{H}_*^\omega \subset S_\omega \mathcal{S}, \quad \mathcal{M}_-^\omega \cap \mathcal{H}_*^\omega \subset S_\omega \mathcal{B}, \quad \mathcal{M}_0^\omega \cap \mathcal{H}_*^\omega \subset S_\omega \mathcal{T}_{C\delta_*}, \quad (7.57)$$

where  $\mathcal{H}_*^\omega$  denotes the constrained region for  $\omega \geq \omega_*$

$$\mathcal{H}_*^\omega := \{\varphi \in H_r^1(\mathbb{R}^3) \mid \mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + c_*\delta_*^2\}. \quad (7.58)$$

Thus we have proven the existence of infinitely many orbits of the 9 cases in the main Theorem 1.1. It remains to see that the manifold extends to the entire set of trapped solutions, together with the threshold property.

**7.3. Extension of the manifold.** Let  $\delta_M \in (0, \delta_X)$  so small that  $\delta_M \ll c_X$  and for any  $\omega \geq \omega_*$  and any  $(\theta, b_+, b_-, \zeta) \in \mathcal{U}_\omega$ ,

$$d_\omega(\mathcal{C}_\omega(\theta, b_+, b_-, \zeta)) < \delta_M \implies |b_+| < \delta_+ \text{ and } \max(|b_-|, \|\zeta\|_\omega) < \delta_-. \quad (7.59)$$

Then the trapping lemma 3.5 implies that

$$\mathcal{M}_1^\omega := \{\varphi \in \mathcal{M}_0^\omega \mid d_\omega(\varphi) < \delta_M, \mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + c_X\delta_M^2\} \quad (7.60)$$

is forward invariant by the flow of (3.2). Let  $\mathcal{M}_2^\omega$  be the maximal backward extension by the flow of this set. Then  $\mathcal{M}_2^\omega$  is the union of all orbits of forward global solutions  $u_\omega$  of (3.2) satisfying

$$\mathbb{A}^\omega(u_\omega) < \mathbb{A}^\omega(Q_\omega) + c_X\delta_M^2, \quad \limsup_{t \rightarrow \infty} d_\omega(u_\omega(t)) < \delta_X. \quad (7.61)$$

Indeed, the trapping lemma automatically improves the last bound to

$$\limsup_{t \rightarrow \infty} d_\omega(u_\omega(t))^2 \leq c_X^{-1}(\mathbb{A}^\omega(u_\omega) - \mathbb{A}^\omega(Q_\omega)) < \delta_M^2 \ll \delta_X^2, \quad (7.62)$$

so  $u_\omega(t)$  belongs to (7.60) for large  $t$ . Hence  $\mathcal{M}_2^\omega$  is a  $C^1$  manifold with codimension 1 and invariant by the flow of (3.2).  $\mathcal{M}_2^\omega$  is unbounded, since it contains solutions that blow up in  $t < 0$ .

To see that  $\mathcal{M}_2^\omega$  is connected, let  $u^\perp$  and  $u^\perp$  be two solutions of (3.2) on  $\mathcal{M}_2^\omega$ . By the above argument, there exists  $T > 0$  such that both  $u^\perp$  and  $u^\perp$  are in  $\mathcal{M}_1^\omega$  for all  $t \geq T$ . Let  $u^\perp(T) = \mathcal{M}_\omega(\theta^\perp, b_-^\perp, \zeta^\perp)$ , then  $\max(|b_-^\perp|, \|\zeta^\perp\|_\omega) < \delta_-$ . For each  $\eta \in [0, 1]$ , let  $u_\eta^\perp$  be the solution of (3.2) with the initial condition

$$u_\eta^\perp(T) = \mathcal{M}_\omega(\theta^\perp, b_-^\perp, \eta\zeta^\perp), \quad (7.63)$$

then  $u_1^\perp = u^\perp$  and  $u_\eta^\perp(T) \in \mathcal{M}_0^\omega$ . Moreover,  $\mathbb{A}^\omega(u_\eta^\perp)$  is decreasing as  $\eta < 1$  decreases until  $\|\eta\zeta^\perp\|_\omega \lesssim |b_-^\perp|^2$ , because

$$\begin{aligned} \eta \frac{d}{d\eta} \mathbb{A}^\omega(u_\eta^\perp(T)) &= -2b_-^\perp b_+' + \langle \mathcal{L}^\omega \eta \zeta^\perp | \eta \zeta^\perp \rangle - \langle N^\omega(v_\eta) | b_+' g_+' + \eta \zeta^\perp \rangle \\ &= \langle \mathcal{L}^\omega \eta \zeta^\perp | \eta \zeta^\perp \rangle + O((|b_-^\perp| + \|\eta \zeta^\perp\|_\omega)^3), \end{aligned} \quad (7.64)$$

where  $b_+' := \partial_\zeta G_\omega(b_-^\perp, \eta \zeta^\perp) \eta \zeta^\perp$  and  $v_\eta := G_\omega(b_-^\perp, \eta \zeta^\perp) g_+' + b_-^\perp g_+' + \zeta^\perp$ . Hence there exists  $\eta_0 \in (0, 1)$  such that  $\mathbb{A}^\omega(u_\eta^\perp)$  is increasing for  $\eta \in [\eta_0, 1]$  and  $\|\eta_0 \zeta^\perp\|_\omega \lesssim |b_-^\perp|^2 \lesssim \delta_M^2$ . Since the energy constraint is preserved, those solutions  $u_\eta^\perp$  are also on  $\mathcal{M}_2^\omega$  for  $\eta \in [\eta_0, 1]$ . In the same way, we obtain a continuous family of solutions  $u_\eta^\perp$  in  $\mathcal{M}_2^\omega$  for  $\eta \in [\eta_1, 1]$  with some  $\eta_1 \in (0, 1)$  such that

$$u_\eta^\perp(T) = \mathcal{M}_\omega(\theta^\perp, b_-^\perp, \eta \zeta^\perp), \quad \|\eta_1 \zeta^\perp\|_\omega \lesssim \delta_M^2. \quad (7.65)$$

Let  $\varphi_s$  be the linear interpolation on  $\mathcal{M}_0^\omega$  between  $u_{\eta_0}^\perp(T)$  and  $u_{\eta_1}^\perp(T)$ , namely

$$\varphi_s := \mathcal{M}_\omega((1-s)\theta^\perp + s\theta^\perp, (1-s)b_-^\perp + sb_-^\perp, (1-s)\eta_0\zeta^\perp + s\eta_1\zeta^\perp) \quad (7.66)$$

for  $s \in [0, 1]$ . Then the same estimate as in (7.64) yields

$$\mathbb{A}^\omega(\varphi_s) - \mathbb{A}^\omega(Q_\omega) \lesssim \delta_M^3 \ll c_X \delta_M^2 \quad (7.67)$$

and so  $\varphi_s \in \mathcal{M}_2^\omega$ . Thus we have obtained a path connecting  $u^\perp(0)$  and  $u^\perp(0)$  in  $\mathcal{M}_2^\omega$ :

$$\begin{aligned} \{u^\perp(t) \mid t : 0 \nearrow T\} \cup \{u_\eta^\perp(T) \mid \eta : 1 \searrow \eta_0\} \cup \{\varphi_s \mid s : 0 \nearrow 1\} \\ \cup \{u_\eta^\perp(T) \mid \eta : \eta_1 \nearrow 1\} \cup \{u^\perp(t) \mid t : T \searrow 0\}. \end{aligned} \quad (7.68)$$

The trapping characterization (7.61) of  $\mathcal{M}_2^\omega$ , together with the distance gap (7.62) from the ejected solutions, implies that for any solution  $u_\omega$  on  $\mathcal{M}_2^\omega$ , the rescaled solution  $u_\beta(t) := S_{\beta/\omega} u_\omega(\omega t/\beta)$  is also on  $\mathcal{M}_2^\beta$  if  $\beta/\omega$  is close enough to 1. Hence rescaling and unifying over  $\omega$  yields a  $C^1$  manifold of codimension 1:

$$\mathcal{M}_3 := \bigcup_{\omega > \omega_*} S_\omega^{-1} \mathcal{M}_2^\omega, \quad (7.69)$$

around  $\mathcal{S}_1|_{\mathbb{M} < \mu_*}$  in  $H_r^1(\mathbb{R}^3)$ . Since  $\mathcal{M}_2^\omega$  is invariant by the rescaled NLS, the above manifold  $\mathcal{M}_3$  is invariant by (1.1) in the original scaling.  $\mathcal{M}_3$  is also connected<sup>1</sup> and unbounded, and it is the union of all orbits of forward global solutions  $u$  of (1.1) such that  $u_\omega := S_\omega u(t/\omega)$  satisfies (7.61) for some  $\omega > \omega_*$ .

<sup>1</sup>Let  $X, Y$  be topological spaces and  $M : X \rightarrow \mathcal{P}(Y)$ . Suppose that  $X$  is connected, and that for every  $x \in X$ ,  $M(x)$  is connected and  $M(x) \cap M(z) \neq \emptyset$  for all  $z$  in a neighborhood of  $x$ . Then  $\bigcup_{x \in X} M(x)$  is also connected.

Restricting  $\omega \geq \omega_*$  and  $\delta_M \leq (c_*/c_X)^{1/2}\delta_*$ , the scattering/blow-up after departure is applicable to the solutions off the manifold. Hence putting

$$\begin{aligned}\mathcal{H}_* &:= \{\varphi \in H_r^1(\mathbb{R}^3) \mid \exists \omega > \omega_*, \mathbb{A}^\omega(S_\omega \varphi) < \mathbb{A}^\omega(Q_\omega) + c_X \delta_M^2\}, \\ \mathcal{M}_* &:= \bigcup_{\omega > \omega_*} S_\omega^{-1} \mathcal{M}_2^\omega\end{aligned}\tag{7.70}$$

we have

$$\mathcal{M}_* = \mathcal{H}_* \cap \mathcal{T}_{C\delta_M}, \quad \mathcal{H}_* \setminus \mathcal{M}_* = \mathcal{H}_* \cap (\mathcal{S} \cup \mathcal{B}).\tag{7.71}$$

Moreover, around each point  $\varphi \in \mathcal{M}_*$ , we can find a small open ball  $B(\varphi) \subset \mathcal{H}_*$  which is separated by  $\mathcal{M}_*$  into  $\mathcal{S}$  and  $\mathcal{B}$ . More precisely,  $B(\varphi) \setminus \mathcal{M}_*$  is open with two connected components  $B^\pm(\varphi)$  such that  $B^+(\varphi) \subset \mathcal{S}$  and  $B^-(\varphi) \subset \mathcal{B}$ . Then  $B_* := \bigcup_{\varphi \in \mathcal{M}_*} B(\varphi)$  is an open neighborhood of  $\mathcal{M}_*$  in  $\mathcal{H}_*$ , separated by  $\mathcal{M}_*$  into two disjoint open sets:  $B_*^\pm := \bigcup_{\varphi \in \mathcal{M}_*} B^\pm(\varphi)$ .  $B_*$  and  $B_*^\pm$  are also connected sets, because  $\mathcal{M}_*$  is<sup>1</sup>.

The time inversion  $\overline{\mathcal{M}_*}$  is a  $C^1$  manifold with codimension 1, consisting of solutions in  $\mathcal{H}_*$  trapped by  $\Psi$  as  $t \rightarrow -\infty$ . Hence  $\mathcal{M}_* \cap \overline{\mathcal{M}_*}$  consists of solutions trapped by  $\Psi$  as  $t \rightarrow +\infty$  and as  $t \rightarrow -\infty$ . The one-pass lemma 5.2 implies that such a solution under the constraint stays within  $O(\delta_M)$  distance in  $H_\omega^1$  around  $\Psi[\omega]$  of some  $\omega > \omega_*$  for all  $t \in \mathbb{R}$ . Hence, taking  $\delta_M \ll \delta_-$ , we have

$$\mathcal{M}_* \cap \overline{\mathcal{M}_*} = \bigcup_{\omega > \omega_*} \{S_\omega^{-1} \varphi \mid \varphi \in \mathcal{M}_0^\omega \cap \overline{\mathcal{M}_0^\omega}, \mathbb{A}^\omega(\varphi) < \mathbb{A}^\omega(Q_\omega) + c_X \delta_M^2\}.\tag{7.72}$$

Actually, for any  $\varphi \in \mathcal{M}_* \cap \overline{\mathcal{M}_*}$  and any  $\omega > \omega_*$  such that  $\mathbb{A}^\omega(S_\omega \varphi) < \mathbb{A}^\omega(Q_\omega) + c_X \delta_M^2$ , we have  $S_\omega \varphi \in \mathcal{M}_0^\omega \cap \overline{\mathcal{M}_0^\omega}$ . Hence  $\mathcal{M}_* \cap \overline{\mathcal{M}_*}$  is a  $C^1$  invariant manifold with codimension 2. The connectedness of  $\mathcal{M}_* \cap \overline{\mathcal{M}_*}$  as well as  $\mathcal{M}_2^\omega \cap \overline{\mathcal{M}_2^\omega}$  is proved in the same way as  $\mathcal{M}_2^\omega$ , namely by reducing the dispersive component  $\zeta$  in the local coordinate on  $\mathcal{M}_0^\omega \cap \overline{\mathcal{M}_0^\omega}$ . The same is for the connectedness of  $\mathcal{S} \cap \overline{\mathcal{M}_*}$  and that of  $\mathcal{B} \cap \overline{\mathcal{M}_*}$ , after rescaling and applying the backward flow in order to send them into the domain of the local coordinate around  $Q_\omega$ .

#### APPENDIX A. TABLE OF NOTATION

symbols	description	defined in
$H = -\Delta + V$	Schrödinger operator with the potential	(1.1), Section 1.5
$e_0, \phi_0$	its eigenvalue and ground state	(1.2)
$Q$	the ground state for NLS	(1.10)
$[\cdot], \mathbb{G}, \mathbb{E}, \mathbb{M}, \mathbb{K}_2$	some functionals	(1.3), (1.32), (1.17), (1.34)
$\mathbb{H}^0, \mathbb{E}^0, \mathbb{A}$	functionals without the potential	(1.32), (1.40)
$\mathbb{A}_\omega, \mathbb{K}_{0,\omega}$	functionals with frequency $\omega$	(1.38)
$\mathbb{E}^\omega, \mathbb{A}^\omega, \mathbb{K}_2^\omega, \mathbb{J}^\omega$	rescaled functionals	(1.38)
$\mathcal{S}, \mathcal{S}_j, \mathcal{E}_j$	all solitons, $j$ -th solitons and energy	(1.5), (1.8), (1.6), (1.7)
$(\Phi, \Omega), \Psi$	the ground and first excited states	(1.11)
$\mu_*, z_*, Z_*, \omega_*$	size of the above coordinates	Lemma 2.6
$H_\omega^1, \ \cdot\ _\omega$	rescaled energy norms	(1.20), (2.79)
$\mathcal{N}_\delta(\Psi), \mathcal{N}_\omega$	neighborhoods of $\Psi$ (unscaled/rescaled)	(1.22), (2.89)
$\mathcal{S}, \mathcal{B}, \mathcal{T}_\delta$	classification of initial data	(1.28)
$L^p, H_p^s, H^s, \dot{H}^s, B_{p,q}^s$	Lebesgue, Sobolev and Besov spaces	Section 1.4



$H_r^s, X_r, L_t^p X(I)$	radial subspaces and $X$ -valued $L^p$ in $t$	Section 1.4
$(\cdot \cdot), \langle \cdot \cdot \rangle$	inner products on $L^2(\mathbb{R}^3)$	Section 1.4
$\text{Stz}^s, \mathfrak{st}$	Strichartz norms	Section 1.4
$\mathcal{S}_p^t, \mathcal{S}_p', S_\omega, V^\omega$	scaling operators and scaled potential	(1.33), (1.39)
$\varphi^\perp, P_\varphi^\perp, P_c$	orthogonal subspace and projection	(1.36), (1.37)
$\triangleleft l(a^\flat) := l(a^\perp) - l(a^\flat)$	difference	Section 1.4
$C_D, C_M, C_S, C_K, \omega_\star$	large constants	Lemmas 2.1, 2.3, 3.3, 5.1
$C_0, C_1$	large constants	(7.20), Lemma 7.2
$\delta_C, \delta_D, \delta_E, c_X, \delta_I, \delta_X$	small constants	Lemmas 2.6, 3.1, 3.2, 3.3,
$\delta_U, \delta_V, \varepsilon_S, \delta_\star, c_\star$	small constants	Lemmas 4.1, 4.2, 5.2, 6.2
$\varepsilon_V, \kappa_V$	small numbers in variational estimates	Lemma 4.1
$\mathcal{L}, \mathcal{L}^\omega, L_\pm, L_\pm^\omega$	linearized operators	(2.23), (2.41)
$Q_\omega, \mathcal{Q}_\omega$	rescaled first excited state and its orbit	(2.30), (2.32)
$Q'_\omega, Q'$	frequency derivatives of solitons	(2.46), (2.50)
$\alpha, \alpha_\omega, g_\pm, g_\pm^\omega$	(un)stable eigenvalues/eigenfunctions	(2.55), (2.67)
$C^\omega(v), N^\omega(v)$	nonlinear part of energy and derivative	(2.70), (3.5)
$P_\star^\omega, \mathcal{P}_\star^\omega, \mathcal{Z}^\omega$	symplectic projections around $Q^\omega$	(2.72), (2.73), (2.91)
$d_{0,\omega}, d_{1,\omega}, d_\omega$	energy-distances to $\mathcal{Q}_\omega$	(2.75), (3.17), (3.19)
$\mathcal{V}_\omega, \mathcal{Z}_\omega$	$H^1$ subspaces with orthogonality	(2.81), (2.82)
$\mathcal{C}_\omega, \mathcal{U}_\omega$	local chart around $\mathcal{Q}_\omega$	(2.87), (2.88)
$m^\omega(v)$	modulation of phase	(3.10)
$\mathcal{N}^\omega(v), \mathcal{N}_\star^\omega(v)$	nonlinearity and its spectral projection	(3.12), (3.13)
$\chi$	smooth cut-off function	(3.16)
$\mathfrak{S}_\omega, \mathfrak{S}$	sign functionals	Lemma 4.2
$I_H, I_V$	sets of hyperbolic and variational times	(5.6)
$\mathcal{V}_m$	localized virial (depending on $\mathfrak{S}_\omega$ )	(5.8), (5.31)
$\mathcal{H}_c[z], R[z]$	subspace and projection around $\mathcal{S}_0$	(5.26)
$B[z]$	linear interaction with $\mathcal{S}_0$	(5.47)
$\text{FS}_\omega(A)$	set of global solutions away from $\mathcal{Q}_\omega$	(6.3)
$\text{ST}^\omega, A_\omega^\star, A^\star$	Strichartz/energy bounds for scattering	(6.5)
$\lambda_n^j, s_n^j, \gamma_n^j$	linear profile, its center and remainder	(6.11), (6.29)
$\xi_\infty^j, \Lambda_n^j, \Gamma_n^j$	nonlinear profiles and remainder	(6.13), (6.32)
$G_\omega, U_\omega, \mathcal{U}_\omega$	the graph of manifold and its domains	Theorem 7.1
$\dot{X}(\vec{v})$	operator for the difference	(7.10)
$\mathcal{M}_0^\omega, \mathcal{M}_\omega$	local manifold and its coordinate	(7.52)

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